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STATIONARITY AND MEMORY OF ARCH(∞) MODELS

PAOLO ZAFFARONI
Banca d'Italia

We establish the necessary and sufficient conditions for covariance stationarity of ARCH(∞), for both the levels and the squares. The result applies to any form of the conditional variance coefficients. This includes GARCH(p,q) and also specifications with hyperbolically decaying coefficients, such as the autoregressive coefficients of the autoregressive fractionally integrated moving average model. The covariance stationarity condition for the levels rules out long memory in the squares.

1. INTRODUCTION

Introduced by the seminal work of Engle (1982), the autoregressive conditional heteroskedastic (ARCH) models are certainly the most popular class of nonlinear time series models, in particular thanks to the development of the generalized autoregressive conditional heteroskedastic model of order p,q (GARCH(p,q)) by Bollerslev (1986) defined by

$$\epsilon_t = z_t \sigma_t, \qquad t \in \mathbb{Z}, \tag{1}$$

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2, \qquad a.s., \tag{2}$$

where $\omega > 0$, $\beta_i \geq 0$, $\alpha_j \geq 0$ $(i = 1,\dots,p,\ j = 1,\dots,q)$ for integers $p \geq 0$ and $q > 0$ and *a.s.* means almost surely. The minimum conditions, which we assume hereafter, imposed on the rescaled innovation z_t are i.i.d.-ness, $|z_t| < \infty$ *a.s.*, and z_t^2 not degenerate. When $p = 0$, one gets the ARCH(q).

Giraitis, Kokoska, and Leipus (2000) find sufficient conditions for strict stationarity of the ARCH(∞). This is the most general formulation of GARCH processes, generalizing (2) to

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$$\sigma_t^2 = \tau + \sum_{k=1}^{\infty} \psi_k \epsilon_{t-k}^2 \quad a.s., \qquad \sum_{k=1}^{\infty} \psi_k < \infty, \tag{3}$$

where $\tau \geq 0$ and $\psi_k \geq 0$. The ARCH(∞) was introduced by Robinson (1991), who considered finite parameterizations of the ψ_k to build classes of alternatives in deriving score tests for no-ARCH. (See also the work of Hong, 1997, who proposes a one-sided test for no-ARCH that employs the ARCH(∞) as a class of alternatives.)

The ARCH(∞) representation of GARCH(p,q) model (2) is obtained choosing exponentially decaying ψ_k; e.g., the GARCH(1,1) follows setting $\tau = \omega/(1-\beta_1)$ and $\psi_j = \alpha_1 \beta_1^{j-1}$; the ARCH($q$) model is obtained when $\tau = \omega$ and $\psi_j = 0$ for $j > q$. Although hyperbolic behavior of the ARCH coefficients ψ_j is allowed for, the strict stationarity condition of Giraitis et al. (2000),

$$E(z_t^2) \sum_{k=1}^{\infty} \psi_k < 1, \tag{4}$$

implies a bounded second moment of the ϵ_t. Indeed, for GARCH(1,1) (4) works out to $E(\beta_1 + \alpha_1 z_t^2) < 1$, the well-known covariance stationarity condition for the ϵ_t (see Bollerslev, 1986), which is more restrictive than the necessary and sufficient strict stationarity condition (see Nelson, 1990).

Exploiting the well-known linear ARMA(m,p) (with $m = \max[p,q]$) representation of GARCH(p,q), introduced by Bollerslev (1986), the autocovariance function (ACF) of the squares ϵ_t^2 can be readily shown to be proportional to the ACF of an ARMA(m,p) once the bounded fourth moment condition of the ϵ_t is imposed. The critical step is precisely calculating this constant of proportionality, given by $E[z_t^2 - E(z_t^2)]^2 E(\sigma_t^4)$. For general GARCH($p,q$) the solution has been found independently by He and Teräsvirta (1999) and Karanasos (1999), extending the GARCH(1,2) and GARCH(2,1) cases analyzed by Bollerslev (1986). Noting that the previous results are either based on un-primitive assumptions or establish only necessity but not sufficiency, Ling (1999) and Ling and McAleer (2002) fill these gaps, providing the necessary and sufficient higher order moment conditions for GARCH(p,q).

This paper shows how to evaluate $E(\sigma_t^4)$ for ARCH(∞) σ_t^2. It uses the linear representation of ϵ_t^2 in a martingale difference sequence. The cases of both exponentially (such as GARCH(p,q)) and hyperbolically decaying coefficients ψ_j are comprised. In particular, necessary and sufficient conditions for weak stationarity of the levels ϵ_t and the squares ϵ_t^2 are established. This is done in Section 2. Finally, we discuss in Section 3 the implications of the covariance stationarity conditions on the memory of the squares. It follows that covariance stationarity of the ϵ_t precludes long memory in the ϵ_t^2. This strengthens the result of Giraitis et al. (2000), who found that a sufficient condition for covariance stationarity of the ϵ_t^2 rules out long memory. The proofs for the results are given in the Appendix.

2. STATIONARITY OF ARCH(∞)

The minimal condition for covariance stationarity of the ϵ_t, is $\kappa = E(z_t^2) < \infty$, by (1). Following Robinson (1991), who considers the case $\kappa = 1$, setting $\psi(L) := 1 - \kappa \sum_{j=1}^{\infty} \psi_j L^j$, we can rewrite (3) as

$$\psi(L)\epsilon_t^2 = \kappa\tau + \nu_t, \tag{5}$$

setting $\nu_t := \epsilon_t^2 - \kappa\sigma_t^2$. By (1) and the i.i.d.-ness of the z_t, $E(\nu_t|\mathcal{F}_{t-1}) = 0$, where $\mathcal{F}_t$ is the σ-field of events induced by the ϵ_s $(s \leq t)$ (for the definition of conditional expectations when the corresponding unconditional expectations may not exist, see Loève, 1978, Sect. 27.2).

Assume that, for complex-valued z, the following "invertibility" condition holds:

$$\exists\ \delta(z) = \sum_{j=0}^{\infty} \delta_j z^j := \psi^{-1}(z), \qquad \delta_0 = 1, \quad \text{s.t.} \sum_{j=0}^{\infty} \delta_j^2 < \infty. \tag{6}$$

A sufficient condition for (6) is

$$|\psi(z)| \neq 0, \qquad |z| \leq 1,$$

implying $\psi(1) > 0$, but we want to allow for the possibility that $\psi(1) = 0$ such as when the ψ_j are the AR(∞) coefficients of the autoregressive fractionally integrated moving average (ARFIMA) filter:

$$\psi(L) = (1 - L)^d \frac{a(L)}{b(L)}, \tag{7}$$

L being the lag operator $(Lz_t = z_{t-1})$ with $d > 0$ (by summability of the ψ_j) and where the finite-order polynomials $a(L)$, $b(L)$ have zeros outside the unit circle in the complex plane. Indeed, note that (6) does not imply absolute summability of the δ_j.

Given (6), (5) can be rewritten as

$$\epsilon_t^2 = \kappa\tau\delta(1) + \sum_{j=0}^{\infty} \delta_j \nu_{t-j}. \tag{8}$$

These simple manipulations suggest that τ, $\kappa < \infty$, and $\kappa \sum_{i=1}^{\infty} \psi_i < 1$ imply covariance stationarity of the ϵ_t. Indeed, it turns out that these are the necessary and sufficient conditions for $E(\epsilon_t^2) < \infty$. This is developed in Theorem 1, where the necessary and sufficient conditions for covariance stationarity of the squares ϵ_t^2 are also established.

THEOREM 1. *Assume that condition (6) holds.*

(i) The necessary and sufficient conditions for $E(\sigma_t^2) < \infty$ *are*

$$\tau < \infty, \tag{9}$$

$$\kappa < \infty, \tag{10}$$

$$\kappa \sum_{i=1}^{\infty} \psi_i < 1. \tag{11}$$

Under these conditions

$$\phi := E(\epsilon_t^2) = \kappa\tau \Big/ \left(1 - \kappa \sum_{i=1}^{\infty} \psi_i\right) < \infty.$$

(ii) The necessary and sufficient conditions for $E(\sigma_t^4) < \infty$ *are*

$$\tau < \infty, \tag{12}$$

$$\theta := E(z_t^2 - \kappa)^2 < \infty, \tag{13}$$

$$\left(\theta \sum_{u=-\infty}^{\infty} \chi_\delta(u) \chi_{\tilde{\psi}}(u)\right) < 1, \tag{14}$$

setting $\tilde{\psi}_0 = 0$, $\tilde{\psi}_k = \psi_k$ $(k \geq 1)$, *and* $\chi_c(u) := \sum_{k=0}^{\infty} c_k c_{k+u}$, $u = 0, \pm 1, \ldots,$ *for any square summable sequence* c_i.

Under these conditions the ϵ_t^2 *are covariance stationary with* ACF

$$\operatorname{cov}(\epsilon_t^2, \epsilon_{t+u}^2) = E(\nu_t^2)\chi_\delta(u), \qquad u = 0, \pm 1, \ldots,$$

where

$$E(\nu_t^2) = \theta(\phi/\kappa)^2 \left(1 - \theta \sum_{u=-\infty}^{\infty} \chi_\delta(u)\chi_{\tilde{\psi}}(u)\right)^{-1} < \infty.$$

Remark 1.1. The ACF of the squares ϵ_t^2 for GARCH(p,q) can be obtained as follows. We first establish the ARCH(∞) representation of GARCH(p,q). Exploiting the ARMA$(\max(p,q),p)$ representation of GARCH(p,q), by standard arguments (see Nelson and Cao, 1992) one obtains

$$\frac{\alpha(L)}{1 - \beta(L)} = \alpha(L)\left(\frac{A_1}{1 - \rho_1 L} + \cdots + \frac{A_p}{1 - \rho_p L}\right),$$

where $\alpha(L) := \alpha_1 L + \cdots + \alpha_q L^q$, $\beta(L) := \beta_1 L + \cdots + \beta_p L^p$, ρ_i $(i = 1, \ldots, p)$ define the inverse of the roots of $|1 - \beta(z)| = 0$, assumed all outside the unit circle in the complex plane and distinct, and

$$A_i := \frac{1}{(1 - \rho_1/\rho_i)\ldots(1 - \rho_{i-1}/\rho_i)(1 - \rho_{i+1}/\rho_i)\ldots(1 - \rho_p/\rho_i)}, \qquad i = 1, \ldots, p,$$

where $A_p = A_1 = 1$ for $p = 1$. The ARCH(∞) coefficients ψ_j of GARCH(p,q) are then

$$\psi_j = \begin{cases} \alpha_1 \mu_{j-1} + \cdots + \alpha_j \mu_0, & j = 1, \ldots, q, \\ \alpha_1 \mu_{j-1} + \cdots + \alpha_q \mu_{j-q}, & j > q, \end{cases} \qquad \textbf{(15)}$$

$$\tau = \omega/(1 - \beta_1 - \cdots - \beta_p),$$

with $\mu_j := A_1(\rho_1)^j + \cdots + A_p(\rho_p)^j$, $j \geq 0$. Based on the previous definitions, Theorem 1 yields the ACF of GARCH(p,q) ϵ_t^2 with

$$\phi = \kappa\omega/(1 - \kappa\alpha(1) - \beta(1)),$$

deriving the $\chi_{\tilde{\psi}}(u)$ and the $\chi_\delta(u)$, and thus $E(\nu_t^2)$, based on (15) and deriving the δ_i from (6).

Remark 1.2. Necessary and sufficient conditions for covariance stationarity of GARCH(p,q) ϵ_t^2 have been independently established by He and Teräsvirta (1999) and Karanasos (1999). When considering GARCH(p,q), these results and (12)–(14) are equivalent, but formal derivation of their equivalence is very cumbersome. He and Teräsvirta (1999) is closer to our result, because their work considers the slightly more general case of nonstandardized z_t, though from a computational point of view our conditions (12)–(14) seem more appealing because they permit immediate use of known results for the theoretical ACF of ARMA(p,q).

As an example, for GARCH(1,1), setting $\kappa = 1$ and $\pi_1 := \alpha_1 + \beta_1$, i.e., the "persistence" parameter, we get

$$\chi_\delta(0) = 1 + \frac{\alpha_1^2}{1 - \pi_1^2}, \qquad \chi_\delta(u) = \alpha_1 \pi_1^{|u|-1}\left(1 + \frac{\alpha_1 \pi_1}{1 - \pi_1^2}\right),$$

$$\chi_{\tilde{\psi}}(0) = \frac{\alpha_1^2}{1 - \beta_1^2}, \qquad \chi_{\tilde{\psi}}(u) = \frac{\alpha_1^2 \beta_1^{|u|}}{1 - \beta_1^2}, \qquad u = \pm 1, \ldots.$$

When the z_t have zero fourth-order cumulant (such as for Gaussian z_t) then $\theta = 2$. Under this condition, by simple manipulations, (14) yields the well-known covariance stationarity conditions for ϵ_t^2 (cf. Bollerslev, 1986, Sect. 3)

$$3\alpha_1^2 + \beta_1^2 + 2\alpha_1\beta_1 < 1, \qquad \textbf{(16)}$$

which, in turn, implies (11): $E(z_t^2)\alpha(1) + \beta(1) < 1$.

For the ARCH(2), namely, $\psi_k = 0$, $k \geq 3$, setting $(1 - \xi_1^{-1}L)(1 - \xi_2^{-1}L) := (1 - \psi_1 L - \psi_2 L^2)$ and assuming that the roots ξ_1, ξ_2 are greater than one in modulus (cf. (6)) and satisfy, for simplicity's sake, $\xi_1 \neq \xi_2$, one gets

$$\chi_\delta(u) = \frac{\xi_1^2 \xi_2^2}{(\xi_1 \xi_2 - 1)(\xi_2 - \xi_1)} ((\xi_1^2 - 1)^{-1} \xi_1^{1-|u|} - (\xi_2^2 - 1)^{-1} \xi_2^{1-|u|}),$$

$$u = 0, \pm 1, \ldots, \qquad \chi_{\tilde{\psi}}(0) = \psi_1^2 + \psi_2^2, \qquad \chi_{\tilde{\psi}}(1) = \psi_1 \psi_2,$$

$$\chi_{\tilde{\psi}}(u) = 0, \qquad u = \pm 2, \pm 3, \ldots,$$

when $\kappa = 1$ (see Brockwell and Davis, 1987, Example 3.3.5). Using

$$\psi_1 = \xi_1^{-1} + \xi_2^{-1}, \qquad \psi_2 = -\xi_1^{-1} \xi_2^{-1},$$

and by simple manipulations, from (14) one gets the covariance stationarity condition for ARCH(2) ϵ_t^2 (cf. Milhøj, 1985, Theorem 3; He and Teräsvirta, 1999, equation (12)), setting $\theta = 2$,

$$3\psi_1^2 + 3\psi_2^2 + 3\psi_1^2 \psi_2 - 3\psi_2^3 + \psi_2 < 1.$$

Remark 1.3. When

$$(E(z_t^4))^{1/2} \sum_{j=1}^{\infty} \psi_j < 1, \tag{17}$$

Giraitis et al. (2000, Theorem 2.1) show that $E(\sigma_t^4) < \infty$. However, (17) is more restrictive than (14). For instance, for GARCH(1,1) with $\theta = 2\kappa = 2$, their condition becomes

$$3^{1/2}\alpha_1 + \beta_1 < 1,$$

strictly implying (16), unless $\beta_1 = 0$, the ARCH(1).

Remark 1.4. Hyperbolically decaying specifications of the ψ_i and hence of the δ_i are allowed for. Note, however, that from (11), $\kappa < 1/(\sum_{i=1}^{\infty} \psi_i)$ and thus $\kappa = 1$ is ruled out when $\sum_{i=1}^{\infty} \psi_i = 1$. Thus, unless $\kappa < 1$, (7) is not compatible with covariance stationary ϵ_t.

Imposing $\kappa = 1$, a choice compatible with covariance stationary levels is obtained using the autoregressive coefficients of the ARFIMA filter as follows. Set (cf. (7))

$$1 - \sum_{j=1}^{\infty} \bar{\psi}_j L^j := (1 - L)^d \frac{a(L)}{b(L)}, \tag{18}$$

where $d > 0$ and $a(L)$ and $b(L)$ are finite-order polynomials, all of whose roots are outside the unit circle in the complex plane. Assuming that the nonnegativity constraints on the coefficients hold, i.e., $\bar{\psi}_i \geq 0$, set $\psi_i = \bar{\psi}_i$ $(i \geq 2)$ and $\psi_1 = \bar{\psi}_1 \epsilon$ for some given $0 < \epsilon < 1$. Condition (14) is more involved, but again by suitable modification of the first n (say) coefficients $\bar{\psi}_j$ $(j = 1, \ldots, n)$, a feasible sequence of coefficients can be obtained from (18).

Remark 1.5. An alternative, frequency domain characterization of (14) is

$$\theta \int_{-\pi}^{\pi} f_\delta(\lambda) f_\psi(\lambda)\, d\lambda < 2\pi, \tag{19}$$

setting

$$f_\delta(\lambda) := |\delta(e^{i\lambda})|^2, \qquad f_\psi(\lambda) := (1/\kappa^2)|1-\psi(e^{i\lambda})|^2, \qquad -\pi \le \lambda < \pi,$$

where $\kappa^2 f_\psi(\lambda) = 1 + f_\delta^{-1}(\lambda) - 2\mathrm{Re}(\delta^{-1}(e^{i\lambda}))$ $(-\pi \le \lambda < \pi)$ and $\mathrm{Re}(\cdot)$ denotes the real part of its argument. This frequency domain specification seems easier to compute than (14). For instance, when considering parameterization (7) of the ψ_j (as for GARCH(p,q)) this allows us to exploit the much greater computational simplicity of rational power spectra compared with their Fourier transforms. This might be relevant when imposing covariance stationarity of the ϵ_t^2 in practical estimation, e.g., when estimating ARCH(∞) using the Whittle estimator (see Giraitis and Robinson, 2001).

The equivalent (19) representation of (14) could be used to show that (14) strictly implies (11). In fact, setting $\theta = 2\kappa = 2$ for simplicity's sake, (19) can be rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\delta(\lambda)(1 - 2\mathrm{Re}(\delta^{-1}(e^{i\lambda})))\, d\lambda < -\frac{1}{2}.$$

However, $f_\delta(\lambda) \ge 0$ $(-\pi \le \lambda < \pi)$, and it is arbitrary, necessarily requiring

$$1 - 2\mathrm{Re}(\delta^{-1}(e^{i\lambda})) = -1 + 2\sum_{j=1}^{\infty} \psi_j \cos(j\lambda) < 0, \qquad -\pi \le \lambda < \pi.$$

Given the nonnegativity of the ψ_i and $\cos(j\lambda) \le 1$, with the equality achieved for $\lambda = 0$ (mod. 2π), this is equivalent to

$$-1 + 2\sum_{j=1}^{\infty} \psi_j < 0,$$

strictly implying $\psi(1) > 0$.

Finally, exploiting the relation between the ψ_i $(i \ge 1)$ and the δ_j $(j \ge 0)$ (cf. (6)), (19) allows us to express the time domain characterization (14) more simply as

$$2\kappa \sum_{j=1}^{\infty} \psi_j \chi_\delta(j) < \frac{\kappa^2}{\theta} + \chi_\delta(0) - 1.$$

Remark 1.6. The original formulation of ARCH(∞), analogous to Robinson (1991) but allowing $\kappa \neq 1$, is

$$\sigma_t^2 = \tilde{\tau} + \sum_{k=1}^{\infty} \psi_k (\epsilon_{t-k}^2 - \kappa \tilde{\tau}), \quad a.s., \tag{20}$$

for some $0 < \tilde{\tau} < \infty$. The reparameterization (20) is clearly permitted only for covariance stationary ϵ_t, i.e., when $\psi(1) > 0$, given that $\tilde{\tau} = \tau/\psi(1)$. In fact, assume that one starts directly from (20) rather than from (3). By the nonnegativity constraint $\kappa \sum_{i=1}^{\infty} \psi_i \leq 1$ because $\psi(1) < 0$ is not allowed. Imposing $\psi(1) = 0$ and assuming (6), the linear moving average representation (8) for the ϵ_t^2 would then be

$$\epsilon_t^2 = \zeta + \sum_{j=0}^{\infty} \delta_j \nu_{t-j},$$

for any constant ζ, given that $\psi(L)\epsilon_t^2 = \psi(L)\epsilon_t^2 - \psi(1)\zeta = \psi(L)(\epsilon_t^2 - \zeta)$, which is meaningless.

Remark 1.7. (9)–(11) and (12)–(14) are the necessary and sufficient conditions for $E(\epsilon_t^2) < \infty$ and for $E(\epsilon_t^4) < \infty$, respectively.

3. MEMORY OF ARCH(∞)

Giraitis et al. (2000, Proposition 3.1) show that (17) implies absolute summability of the ACF for the ϵ_t^2, ruling out long memory. However, considering that (17) is more restrictive than effectively required to obtain covariance stationary ϵ_t^2, it is important to assess the impact of the weaker condition (14) on the memory of the ϵ_t^2.

Insights can be obtained by looking at the linear representation (8) for ϵ_t^2. In fact, it follows that the memory of the ϵ_t^2 is expressed by the asymptotic behavior of δ_j as $j \to \infty$. Surprisingly, it turns out that even the much weaker covariance stationarity condition (11) for the levels ϵ_t rules out long memory in the ϵ_t^2. In fact, from (6) and (11),

$$\delta(1) = \sum_{j=0}^{\infty} \delta_j = 1 \Bigg/ \left(1 - \kappa \sum_{j=1}^{\infty} \psi_j\right) < \infty.$$

Expression (14) ensures that the uncorrelated ν_t have finite variance but the rate of decay of the δ_j, imposed by (11), is already quick enough to imply their absolute summability. We summarize our results on the memory of the ϵ_t^2 as follows.

THEOREM 2. *Assume that conditions (6) and (11) hold. Then*

$$\sum_{j=1}^{\infty} \delta_j < \infty, \tag{21}$$

where

$$0 \le \delta_l = \kappa\psi_l + \sum_{s=2}^{l} \kappa^s \sum_{i_1=1}^{l-s+1} \cdots \sum_{i_{s-1}=1}^{l-i_1-\cdots-i_{s-2}-1} \psi_{i_1}\ldots\psi_{i_{s-1}}\psi_{l-i_1-\cdots-i_{s-1}}, \qquad l \ge 1. \tag{22}$$

When the ψ_i *decay toward zero more slowly than exponentially, namely,* $\psi_i/\zeta^i \to \infty$ *as* $i \to \infty$ *for any* $0 < \zeta < 1$, *(21) implies that, as* $u \to \infty$,

$$\chi_\delta(u) \sim C\psi_u, \tag{23}$$

for some $0 < C < \infty$, *with* $c(x) \sim d(x)$ *as* $x \to x_0$, *meaning that* $c(x)/d(x) \to 1$.

Remark 2.1. When $\psi_i \sim ci^{-\delta}$ as $i \to \infty$ for $0 < c, \delta - 1 < \infty$, as for the parameterization described in Remark 1.4, $E(\epsilon_t^2) < \infty$ implies

$$\chi_\delta(u) \sim Cu^{-\delta}, \qquad u \to \infty, \tag{24}$$

for some $0 < C < \infty$, ruling out long memory in the ϵ_t^2. Under the same assumptions on the asymptotic behavior of the ψ_i, the exact rate in (24) was also obtained in Giraitis et al. (2000, Proposition 3.2), although they impose (17), a sufficient condition for $E(\epsilon_t^4) < \infty$.

Remark 2.2. Theorem 2 makes it clear that whereas the bounded second moment conditions (9)–(11) impart the degree of memory of the ϵ_t^2, the stronger bounded fourth moment conditions (12)–(14) ensure that the martingale difference sequence ν_t (the innovations in the linear representation (8) of the squares) are square integrable, without changing the memory implications of the model. This double role of the coefficients is simply a by-product of ARCH(∞) nonlinearity.

Remark 2.3. In the case of exponentially decaying ψ_i, i.e., when there does exist $0 < \zeta < 1$ such that $\psi_i/\zeta^i \to c$ as $i \to \infty$ for some $0 < c < \infty$, it clearly follows that

$$\chi_\delta(u) \sim C\delta^u, \qquad u \to \infty,$$

for some $0 < \delta < 1$, $0 < C < \infty$, as in the GARCH(p,q) case.

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APPENDIX

Proof of Theorem 1. Part (i). Assume that $E(\sigma_t^2) < \infty$. The necessary condition of (9)–(11) follows, taking expectation on both sides of (3). Sufficiency of (9)–(11) for $E(\sigma_t^2) < \infty$ is proved in Giraitis et al. (2000, Theorem 2.1). Part (ii). Assume that $E(\sigma_t^4) < \infty$. Then (12)–(14) follow, squaring both terms in (3) and taking expectations,

$$E(\sigma_t^4) = (\phi/\kappa)^2 + \theta E(\sigma_t^4) \sum_{j_1,j_2=1}^{\infty} \psi_{j_1}\psi_{j_2}\left(\sum_{i=0}^{\infty} \delta_i \delta_{i+(j_1-j_2)}\right),$$

using

$$E[(\epsilon_t^2 - \phi)(\epsilon_{t+u}^2 - \phi)] = E(\nu_t^2) \sum_{j=0}^{\infty} \delta_j \delta_{j+u}, \qquad u = 0, \pm 1, \ldots,$$

as the ν_t are square integrable martingale differences and $\sum_{i=0}^{\infty} \delta_i^2 < \infty$ by (6). Given (12)–(14), $E(\epsilon_t^2) = \phi < \infty$ by part (i). Therefore ϵ_t is strictly stationary. Next, setting $1_t := 1(\bigcap_{j=0}^{\infty}\{\sigma_{t-j}^2 < M\})$ for some constant $M < \infty$ where $1(A)$ equals one when the event A holds and zero otherwise, for any arbitrary sequence of constants $s_j \geq 0$, $j = 1, 2, \ldots,$

$$\sigma_t^2 1_t \leq \tau 1_t + \sum_{j=1}^{\infty} \psi_j \epsilon_{t-j}^2 1_{t-j-s_j}.$$

Writing $\epsilon_{t-j}^2 = \epsilon_{t-j}^2 - \phi + \phi$, using (8), squaring and rearranging terms yields, as $M \to \infty$,

$$\begin{aligned}\sigma_t^4 1_t \leq{}& A_t^2 + 2A_t \sum_{j=1}^{\infty} \psi_j (\epsilon_{t-j}^2 - \phi) 1_{t-j-s_j} \\ &+ \sum_{k_1=0}^{\infty} \sum_{j_1,j_2=1}^{\infty} \psi_{j_1}\psi_{j_2}\delta_{k_1}\delta_{k_1+j_1-j_2} 1_{t-j_1-s_{j_1}} \nu_{t-k_1-j_1}^2 \\ &+ \sum_{k_1,k_2=0}^{\infty} \delta_{k_1}\delta_{k_2} \sum_{\substack{j_1,j_2=1 \\ j_1+k_1 \neq j_2+k_2}}^{\infty} \psi_{j_1}\psi_{j_2} 1_{t-j_1-s_{j_1}} 1_{t-j_2-s_{j_2}} \nu_{t-k_1-j_1}\nu_{t-k_2-j_2}, \end{aligned} \tag{A.1}$$

setting $A_t := \tau 1_t + \phi \sum_{j=1}^{\infty} \psi_j 1_{t-j-s_j}$. For the expectation of the first two terms on the right-hand side of (A.1), as $M \to \infty$,

$$EA_t^2 \to (\phi/\kappa)^2,$$

and

$$\sum_{j=1}^{\infty} \psi_j EA_t(\epsilon_{t-j}^2 - \phi) 1_{t-j-s_j} \to 0.$$

By the law of iterated expectations, setting $P(A) = E1(A)$ for any event A,

$$\sigma_{t-k}^4 1_t = E(P(\sigma_t^4 < M, \ldots, \sigma_{t-k+1}^4 < M | \sigma_{t-k}^4, \ldots) 1_{t-k}\sigma_{t-k}^4) \leq E(\sigma_{t-k}^4 1_{t-k}) = E(\sigma_t^4 1_t),$$

the latter equality holding by strict stationarity. Finally, setting $s_{j_1} = k_1$, $s_{j_2} = k_2$, taking expectations and collecting terms in (A.1) yields

$$E(\sigma_t^4 1_t) \leq E(A_t^2)\left(1 - \theta \sum_{u=-\infty}^{\infty} \chi_\delta(u)\chi_{\tilde{\psi}}(u)\right)^{-1} + o(1). \qquad \blacksquare$$

Proof of Theorem 2. From

$$\left(1 + \sum_{j=1}^{\infty} \delta_j L^j\right)\left(1 - \kappa \sum_{i=1}^{\infty} \psi_i L^i\right) = 1,$$

by the fundamental theorem for polynomials and simple yet tedious calculations, (22) follows. We now establish that

$$\delta_j = \kappa\psi_j + O\left(\left(\kappa \sum_{i=1}^{\infty} \psi_i\right)^{\zeta j}\right), \quad \text{as } j \to \infty, \tag{A.2}$$

for some constant $0 < \zeta < \infty$. By (11), for any $M < \infty$

$$\kappa \sum_{j=1}^{M} \psi_j < 1, \qquad \kappa \sum_{j=M+1}^{\infty} \psi_j < 1.$$

Consider a large enough k such that $M < k$ for one of such M. Then, split the right-hand side of (22) as

$$\delta_k = \kappa\psi_k + \sum_{l=2}^{[k/M]} \sum_{j_1=1}^{k-l+1} \cdots \sum_{j_{l-1}=1}^{k-j_1-\cdots-j_{l-2}-1} B_{j_1,j_2,\ldots,j_{l-1}}(l,k)$$
$$+ \sum_{l=[k/M]+1}^{k} \sum_{j_1=1}^{k-l+1} \cdots \sum_{j_{l-1}=1}^{k-j_1-\cdots-j_{l-2}-1} B_{j_1,j_2,\ldots,j_{l-1}}(l,k), \quad \textbf{(A.3)}$$

where $[\cdot]$ is the integer part of its argument and

$$B_{j_1,j_2,\ldots,j_{l-1}}(l,k) := \kappa^{l-1}\psi_{j_1}\psi_{j_2}\ldots\psi_{j_{l-1}}\psi_{k-j_1-\cdots-j_{l-2}}. \quad \textbf{(A.4)}$$

Let us dispose of the first sum on the right-hand side of (A.3). As $l \le [k/M]$, one obtains three terms:

$$\sum_{l=2}^{[k/M]} \sum_{j_1=1}^{k-l+1} \cdots \sum_{j_{l-1}=1}^{k-j_1-\cdots-j_{l-2}-1} B_{j_1,j_2,\ldots,j_{l-1}}(l,k)$$
$$= \sum_{l=1}^{[k/M]} \left\{ \left(\sum_{j_1=1}^{M} + \sum_{j_1=M+1}^{k-l+1} \right) \ldots \left(\sum_{j_l=1}^{M} + \sum_{j_{l-1}=M+1}^{k-j_1-\ldots-j_{l-2}-1} \right) B_{j_1,j_2,\ldots,j_{l-1}}(l,k) \right\}$$
$$= A_1 + A_2 + A_3 \quad \textbf{(A.5)}$$

with

$$A_1 := \sum_{l=2}^{[k/M]} \left\{ \sum_{j_1=1}^{M} \ldots \sum_{j_{l-1}=1}^{M} B_{j_1,j_2,\ldots,j_{l-1}}(l,k) \right\}, \quad \textbf{(A.6)}$$

$$A_2 := \sum_{l=2}^{[k/M]} \left\{ \sum_{j_1=M+1}^{k-l+1} \ldots \sum_{j_{l-1}=M+1}^{k-j_1-\cdots-j_{l-2}-1} B_{j_1,j_2,\ldots,j_{l-1}}(l,k) \right\}, \quad \textbf{(A.7)}$$

and the "mixed" term

$$A_3 := \sum_{l=2}^{[k/M]} \sum_{s=1}^{l-1} \widetilde{\sum_{l,s}} \left\{ \sum_{j_{i_1}=M+1}^{k-l+i_1-j_1-\cdots-j_{i_1-1}} \cdots \sum_{j_{i_{l-s}}=M+1}^{k-l+i_{l-s}-j_1-\cdots-j_{i_{l-s}-1}} \sum_{j_{r_1}=1}^{M} \cdots \sum_{j_{r_s}=1}^{M} B_{j_1,j_2,\ldots,j_{l-1}}(l,k) \right\},$$
$$\textbf{(A.8)}$$

where $\widetilde{\sum}_{l,s} := \sum_{\{r_1,\ldots,r_s\}\cup\{i_1,\ldots,i_{l-s}\}=\{1,\ldots,l\}}^{\{r_1,\ldots,r_s\},\subset\{1,\ldots,l\}}$, i.e., it selects all the groups of indexes of dimension s (with $s = 1,\ldots,l-1$) drawn from a number l of them and thus for any s and any permutation $j_{i_1} + \cdots + j_{i_{l-s}} + j_{r_1} + \cdots + j_{r_s} = j_1 + \cdots + j_l$. Note that $\widetilde{\sum}_{l,s} 1 = \binom{l}{s}$.

For A_1, for some $0 < \zeta < 1$, writing $\sum_{l=2}^{[k/M]} = \sum_{l=2}^{[\zeta k/M]-1} + \sum_{l=[\zeta k/M]}^{[k/M]}$ yields, as $k \to \infty$,

$$A_1 = O\left(\psi_{[(1-\zeta)k]} X_1 + \left(\kappa \sum_{j=1}^{M} \psi_j\right)^{[\zeta k/M]}\right)$$

$$= O\left(\psi_k + \left(\kappa \sum_{j=1}^{\infty} \psi_j\right)^{[\zeta k/M]}\right)$$

since

$$\sum_{l=0}^{\infty}\left(\sum_{j_1=1}^{M} \dots \sum_{j_l=1}^{M} \psi_{j_1}\psi_{j_2}\dots\psi_{j_l}\kappa^l\right) \le \frac{1}{1-\kappa\sum_{j=1}^{\infty}\psi_j} := X_1 < \infty.$$

Concerning A_2, along the same lines, as $k \to \infty$,

$$A_2 = O\left(\psi_{[(1-\zeta)k]} \sum_{l=0}^{\infty}\left(\sum_{j_1=M+1}^{\infty} \dots \sum_{j_{l-1}=M+1}^{\infty} \kappa^l \psi_{j_1}\psi_{j_2}\dots\psi_{j_{l-1}}\right) + \sum_{l=[\zeta k/M]}^{[k/M]}\left(\kappa \sum_{j=M+1}^{\infty} \psi_j\right)^l\right)$$

$$= O\left(\psi_k + \left(\kappa \sum_{j=1}^{\infty} \psi_j\right)^{[\zeta k/M]}\right).$$

For the "mixed" term, by means of the same truncation, one obtains $A_3 = A_3' + A_3''$ where $A_3' = O(\psi_k X_2) = O(\psi_k)$ as $k \to \infty$, setting

$$X_2 := \sum_{l=0}^{\infty}\sum_{s=1}^{l-1}\widetilde{\sum}_{l,s}\left\{\sum_{j_{i_1}=M+1}^{\infty} \dots \sum_{j_{i_{l-s}}=M+1}^{\infty} \sum_{j_{r_1}=1}^{M} \dots \sum_{j_{r_s}=1}^{M} \psi_{j_1}\psi_{j_2}\dots\psi_{j_l}\kappa^l\right\}$$

$$= \sum_{l=0}^{\infty}\left(\kappa\sum_{j=1}^{\infty}\psi_j\right)^l < \infty.$$

Likewise

$$A_3'' = O\left(\sum_{l=[\zeta k/M]}^{[k/M]}\sum_{s=1}^{l-1}\widetilde{\sum}_{l,s}\sum_{j_{i_1}=M+1}^{\infty} \dots \sum_{j_{i_{l-s}}=M+1}^{\infty} \sum_{j_{r_1}=1}^{M} \dots \sum_{j_{r_s}=1}^{M} B_{j_1,j_2,\dots,j_{l-1}}(l,k)\right)$$

$$= O\left(\left(\kappa\sum_{j=1}^{\infty}\psi_j\right)^{[\zeta k/M]}\right) \quad \text{as } k \to \infty.$$

For the second term on the right-hand side of (A.3), $l > [k/M]$ implies that for at least one summation, say, $\sum_{j_i=1}^{k-l+i-j_1-\dots-j_{i-1}}$, then $k - l + i - j_1 - \dots - j_{i-1} < M$. Thus, decomposition (A.5) is not feasible. However, the previous bounds developed for the case $[\zeta k/M] \le l \le [k/M]$ can still be applied. In fact

$$\sum_{l=[k/M]+1}^{k} \sum_{j_1=1}^{k-l+1} \cdots \sum_{j_l=1}^{k-j_1-\cdots-j_{l-1}} B_{j_1,j_2,\ldots,j_{l-1}}(l,k)$$

$$= O\left(\sum_{l=[k/M]+1}^{k} \sum_{j_1=1}^{(M+1)k-l+1} \cdots \sum_{j_l=1}^{(M+1)k-j_1-\cdots-j_{l-1}} \kappa^l \psi_{j_1} \ldots \psi_{j_l} \right), \qquad \textbf{(A.9)}$$

given $\max_{k\geq 1} \psi_k < \infty$ and $\psi_k \geq 0$ for any k. Then, apply (A.5) to the expression on the right-hand side of (A.9) and proceed as indicated previously.

Condition (11) ensures summability of the δ_j. On the other hand when $\kappa \sum_{i=1}^{\infty} \psi_i = 1$, (A.2) is loose, suggesting a rate of decay of δ_j slower than ψ_j. For instance, for case (7) with $\kappa = 1$ and $d > 0$ it follows that $\psi_j \sim cj^{-d-1}$ and $\delta_j \sim c' j^{d-1}$ as $j \to \infty$ for some $0 < c, c' < \infty$. Finally, note that when $\psi_j \geq 0$ $(j \geq 1)$ then (22) implies $\delta_j \geq 0$ $(j \geq 0)$.

Hence, for any $u > 0$ and some $0 < c < \infty$,

$$\chi_\delta(u) = \sum_{j=0}^{u} \delta_j \delta_{j+u} + \sum_{j=u+1}^{\infty} \delta_j \delta_{j+u} \sim c\delta_u (1 + o(1)), \quad \text{as } u \to \infty,$$

given that, for large enough u,

$$\sum_{j=u+1}^{\infty} \delta_j \delta_{j+u} \leq \delta_{u+n} \sum_{j=u+1}^{\infty} \delta_j = o(\delta_u),$$

for some constant $1 \leq n < \infty$. Finally, comparing (A.2) with (22) and given $\kappa > 0, \psi_i \geq 0$ $(i \geq 1)$, yields $\delta_k \sim c\, \psi_k$ as $k \to \infty$. ■