Permanent and Transitory Components in a Continuous Time Model of the Term Structure

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Abstract: - We develop a new continuous time term structure model which assumes that the instantaneous interest rate is generated by the joint effect of a stationary component which induces temporary effects, and hence mean reversion in the interest rate, and a non stationary component which induces permanent effects and accounts for the martingale behaviour of the riskless rate. We provide a closed form solution for the equilibrium yield curve in the special case where the interest rate is given by a mixture of autoregressive and random walk processes. Parametric return autocorrelation tests and non parametric variance ratio tests show that the US short-term interest rate is a unit root process with significant mean reverting component.

Key-words: - Term structure, brownian motion, mean reversion, random walk, variance ratio.

1 Introduction

The pricing and analysis of the dynamic behaviour of bonds and interest rate sensitive securities require the specification of the processes that drive the underlying sources of uncertainty. Continuous time single-factor term structure models (e.g., Merton (1973,1974), Brennan and Schwartz (1977), Vasicek (1977), Dothan (1978), Cox, Ingersoll and Ross (1985)), or two-factor term structure models (e.g., Richard (1978), Brennan and Schwartz (1982, 1983), Longstaff and Schwartz (1992)) provide a natural, rich and parsimonious framework for modelling the dynamic evolution of the entire yield curve. Stochastic models of the term structure of this kind have been used during the last two decades for trading, hedging and general risk management of the resulting positions over the wide range of interest rate related securities. A common feature to all term structure models is that the instantaneous interest rate is either the sole state variable or one of the two (or more) driving factors of the yield curve.

The default - free short term interest rate is also a key economic variable, since it is an important input for business cycle analysis through its impact on the cost of credit, its sensitivity to the stance of monetary policy and to inflationary expectations. Given the importance of the specific interest rate process in full-fledged macroeconomic models and for term structure modelling, it would seem inappropriate and potentially risky for confident risk management to select an interest rate diffusion that is not congruent with the data.

Our theoretical model assumes that the instantaneous interest rate is generated by the joint effect of two unobserved components: a stationary component which induces temporary effects and hence mean reversion in the interest rate,[1] and a nonstationary component which induces permanent effects and accounts for the martingale behaviour of the riskless rate.[2] We proceed by deriving the valuation partial differential equation that, in equilibrium, all discount bond prices must satisfy along with its probabilistic solution. We take one step further to provide a closed-form solution for the equilibrium yield curve when the temporary component is modelled as an Ornstein-Uhlenbeck process and the permanent component is modelled as an Arithmetic Brownian motion process. In other words, in the spirit of the Fama and French (1988) discrete time model, we are superimposing Vasicek's (1977) mean-reverting Gaussian term structure model on Merton's (1973) random walk framework. We test the significance of the permanent and transitory components in the US interest rate using parametric returns autocorrelation tests, proposed by Fama and French (1988), and non-parametric variance ratio tests, suggested by Cochrane (1988).

The remainder of this paper is organized as follows. In Section 2, we present our continuous term structure model, in which the instantaneous interest rate is generated by a mix of nonstationary and stationary unobserved components. In Section 3, we present our empirical results. Section 4 concludes the paper.

2 A Simple Term Structure Model

Let $P(t,\tau)$ be the price as of calendar time t of a discount bond maturing at time $\tau = t+T, T \geq 0$, with unit maturity value, i.e., $P(\tau,\tau) = 1$. The yield-to-maturity, $R(t,\tau)$, at time t for a bond maturing at time τ can be defined, given $P(t,\tau)$, as the steady state at which the price should increase if the bond is to be worth one currency unit at time τ. It then follows that $R(t,\tau) = -\frac{1}{T}\log P(t,\tau)$. The spot interest rate is defined as $r_t = R(t,t)$. Following the lead of Vasicek (1977), we proceed with the following assumptions.

Assumption A1: As in Fama and French (1988), the spot interest rate r_t is modelled as the sum of a permanent (nonstationary) component q_t, and a temporary (stationary) component z_t which follow unobserved continuous Markov processes:

$$r_t = q_t + z_t, \tag{1}$$

where

$$dq_t = f^{(q)}(r,t)\,dt + \rho^{(q)}(r,t)\,dW^{(q)}, \tag{2}$$

$$dz_t = f^{(z)}(r,t)\,dt + \rho^{(z)}(r,t)\,dW^{(z)}. \tag{3}$$

Both the drift, $f^{(q)}(r,t)$, $f^{(z)}(r,t)$, and diffusion functions, $\rho^{(q)}(r,t)$, $\rho^{(z)}(r,t)$, are sufficiently well behaved for an application of Ito's Lemma (see Arnold (1974)). In general, the Wiener processes $W^{(q)}$ and $W^{(z)}$ will be uncorrelated. We denote the instantaneous covariance matrix by:

$$\sum = \begin{bmatrix} \rho^{(q)^2} & 0 \\ 0 & \rho^{(z)^2} \end{bmatrix}. \tag{4}$$

Applying the two-dimensional Ito's Lemma, using expressions (1)-(4), we obtain the stochastic law of motion of the spot rate in terms of its unobserved components:

$$dr_t = dq_t + dz_t. \tag{5}$$

Assumption A2: The price $P(r,t,\tau)$ of a discount bond is determined by the assessment at time t, of the segment $\{r_s,\ t \leq s \leq \tau\}$ of the spot rate process over the term of the bond.

Assumption A3: The market is efficient, there are no transaction costs, information is available to all investors simultaneously and every investor acts rationally.

Assumptions A1, A2, and A3 imply that the magnitude of the spot rate is the only determinant of the whole term structure and expectations formed

[1] It is well known that one of the most distinguished patterns of any interest rate process is that high (low) values of rates, in historical terms, tend to be followed by a decrease (increase) in rates more frequently than by an increase (decrease).

[2] Generally, standard unit root tests indicate that interest rates follow non-stationary stochastic processes.

with the knowledge of all past developments (including the present) are equivalent to expectations conditional only on the present value of the spot rate. Furthermore, the current value of the spot rate is given by the interaction of two unobserved independent stochastic components, one causing temporary effects while the other causes transitory effects.

Proposition 1.[3] *Under assumptions A1, A2, and A3, the price $P(r,t,\tau)$ of a discount bond at time t of maturity $\tau = t + T$, given the state variable r_t is given by:*[4]

$$P(r,t,\tau) = E_t\{\exp[-\int_t^{\tau} (q_s + z_s)\,ds - \frac{1}{2}\int_t^{\tau}\left(\phi^{(q)^2} + \phi^{(q)^2}\right)ds + \int_t^{\tau}\phi^{(q)}dW^{(q)} + \int_t^{\tau}\phi^{(z)}dW^{(z)}]\}. \quad (6)$$

The fundamental partial differential equation for pricing discount bonds in a market characterized by assumptions A1, A2, A3 is

$$\frac{1}{2}\left(\rho^{(q)^2}P_{qq} + \rho^{(z)^2}P_{zz}\right) + \left(f^{(q)}P_q + f^{(z)}P_z\right) + P_t - rP + P_q\rho^{(q)}\phi^{(q)} + P_z\rho^{(z)}\phi^{(z)} = 0. \quad (7)$$

(Note that in the above expression subscripts denote partial derivatives.)

To illustrate the general model, the term structure of interest rates will now be obtained explicitly in the situation characterized by the following assumptions.

Assumption A4: The temporary component of the spot interest rate follows the Ornstein-Uhlenbeck process:

$$dz_t = \alpha(\gamma - z_t)\,dt + \rho^{(z)}dW^{(z)}, \quad (8)$$

where α is the speed-of-adjustment coefficient $(\alpha > 0)$, γ is the long run mean of the process, and $\rho^{(z)}$ is the diffusion coefficient which allows the process to fluctuate around its long run mean in a continuous but erratic way. The interest rate diffusion in expression (8) is also known as an elastic random walk; it is both Gaussian and Markovian and it was used by Vasicek (1977) in his celebrated term structure model. The permanent component of the spot interest rate follows an Arithmetic Brownian Motion process

$$dq_t = \mu dt + \rho^{(q)}dW^{(q)}, \quad (9)$$

where μ and $\rho^{(q)}$ are constants. This parameterization of the spot rate dynamics was used by Merton (1973). Finally, $dW^{(z)}$ and $dW^{(q)}$ are assumed to be independent (standard) Wiener processes.

Assumption A5: The two market prices of risk, $\phi^{(q)}$ and $\phi^{(z)}$, are constant, i.e., independent of the calendar time and the level of the temporary and permanent components.

The following Theorem is the key result in our paper.

Theorem 1. *Under assumptions A1-A5, the solution of the term structure equation (6) is:*

$$\ln[P(r,t,\tau)] = [R(\infty) - z_t]\left(\frac{1}{\alpha}\right)\left(1 - e^{-\alpha T}\right) - R(\infty)T - \left(\frac{\rho^{(z)^2}}{4\alpha^3}\right)\left(1 - e^{\alpha T}\right)^2 - q_tT - \frac{1}{2}\left(\mu + \rho^{(q)}\phi^{(q)}\right)T^2 + \frac{1}{6}\rho^{(q)^2}T^3, \quad (10)$$

where

$$R(\infty) = \gamma + \frac{\rho^{(z)}\phi^{(z)}}{\alpha} - \frac{1}{2}\left(\frac{\rho^{(z)^2}}{\alpha^2}\right). \quad (11)$$

As in Vasicek (1977), $R(\infty)$ is the yield-to-maturity of a consol bond, where the interest rate follows the Ornstein-Uhlenbeck process in (8) alone.

Proof.

The proof of Theorem 1 is based on a guess solution. Under the Ornstein-Uhlenbeck process in (8) and the Arithmetic Brownian Motion process in (9),

[3] The proof is available upon request.

[4] Note that in the above expression we suppress functional dependencies for notational brevity.

the valuation partial differential equation (PDE) in (7) can be written as:

$$\frac{1}{2}\rho^{(q)^2}P_{qq} + \frac{1}{2}\rho^{(z)^2}P_{zz} + \mu P_q + \alpha(\gamma - z)P_z + P_t - (q+z)P - P_q\rho^{(q)}\phi^{(q)} - P_z\rho^{(z)}\phi^{(z)} = 0. \quad (12)$$

To determine a solution to the PDE (12), let

$$P(r,t,\tau) = A(t,\tau)e^{-B(t,\tau)z}e^{-qT+F(t,\tau)}. \quad (13)$$

Substituting the price equation (13) and its partial derivatives (P_z, P_{zz}, P_q, P_{qq} and P_t) into the valuation PDE (12), gives:

$$\frac{1}{2}\rho^{(q)^2}T^2 - \mu T + F_t - \rho^{(q)}\phi^{(q)}T = 0, \quad (14)$$

$$\frac{1}{2}\rho^{(z)^2}B^2A - \alpha\gamma BA + A_t - \rho^{(z)}\phi^{(z)}BA = 0, \quad (15)$$

$$\alpha B - B_t - 1 = 0, \quad (16)$$

with boundary conditions

$$F(\tau,\tau) = 0, A(\tau,\tau) = 1, B(\tau,\tau) = 0. \quad (17)$$

It follows that if (14)-(16) are solved subject to the boundary conditions (17), the valuation PDE (12) provides the price of a discount bond maturing at time $\tau = t + T$. The solution to (14) is

$$F(t,\tau) = \frac{1}{6}\rho^{(q)^2}T^3 - \left(\phi^{(q)}\rho^{(q)} + \mu\right)\left(\frac{T^2}{2}\right). \quad (18)$$

Solving (15) and (16) leads to a formula similar to Vasicek's (1977):

$$B(t,\tau) = \frac{1 - e^{-\alpha T}}{\alpha}, \quad (19)$$

$$A(t,\tau) = \exp\left[\frac{(B(t,\tau)-T)\left(\alpha^2\gamma + \alpha\rho^{(z)}\phi^{(z)} - \left(\rho^{(z)^2}/2\right)\right)}{\left(\alpha^2 - \left(\rho^{(z)^2}B(t,\tau)^2/4\alpha\right)\right)}\right]. \quad (20)$$

Substituting (18), (19) and (20) into (13), and rearranging, gives (10) in Theorem 1.

3 Empirical Results

In this section we test the significance of the permanent and the temporary components in the short term interest rate using both parametric and non parametric tests. As is standard practice in related empirical work, an observed interest rate series will be used as a proxy of the latent variable, i.e. the instantaneous interest rate, that drives the entire term structure.

We use monthly time series data for the US 3-month Tbill rate covering the period 1/1972 - 8/1997. Application of the Dickey-Fuller and Phillips-Perron unit root tests indicates that the US Tbill rate follows an integrated of order one, $I(1)$, process. However, it is widely known that these tests have very low power.

Cochrane (1988) uses the Beveridge and Nelson (1981) decomposition to express a first-difference stationary process as the sum of (covariance) stationary and random walk components. He argues that a measurement of the size of the random walk component can be a better guide to the proper statistical characterization of the series than a simple unit root test. He proposes a non-parametric method, the Variance Ratio, for determining the magnitudes of the random walk and stationary components of a time series.[5]

The Variance Ratio statistic can be estimated as follows:[6]

$$\widehat{VR}_k = 1 + 2\sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)\widehat{\rho}_j, \quad (21)$$

where $\widehat{\rho}_j$ is the j-th sample autocorrelation coefficient of the first difference of the series, Δr_t.

Estimates of $\widehat{VR}_k$ close to zero (one) indicate that the underlying stochastic process is stationary (a random walk). Values between zero and one indicate that the series contains both random walk and stationary components. Stated differently, there is

[5] For example, if the interest rate r_t follows an ARIMA(0,1,1) stochastic process with a moving average parameter equal to -0.6, the Variance Ratio statistic shows that the permanent component accounts for only 12% of the actual change in r_t, in the long-run.

[6] See, for example, Huizinga (1987), and Campbell and Mankiw (1988).

evidence for mean reversion when $\widehat{VR}_k$ stabilizes below unity as k increases; this implies that an increase in the level of the current interest rate will be reversed by decreases in the future.

Fig. 1 plots the Variance Ratio Statistic and shows that it stabilizes below 1.0. In particular, the variance of the change in the permanent component of the US nominal interest rate is roughly 50% of the variance of its actual change. Thus, the US Tbill rate has a significant mean reverting component.

The size of the stationary component of an $I(1)$ series can also be measured by a regression procedure which involves estimation of the serial correlation of interest rate changes over various horizons.[7] We estimate the k-th autocorrelation of the change in the interest rate over k periods by the slope coefficient (β_k) in the following regression:

$$r_{t+k} - r_t = \alpha_k + \beta_k (r_t - r_{t-k}) + \varepsilon_{k,t}. \qquad (22)$$

When β_k is zero then the behaviour of the interest rate is consistent with that predicted by a random walk model. Negative (positive) values of β_k provide evidence for (against) mean reversion.

When nominal interest rates have both random walk and slowly decaying stationary components, the plot of β_k as a function of k might be U-shaped: β_k is close to zero at short horizons (small k) as the slowly decaying stationary component does not allow mean reversion to manifest itself; as k increases the temporary component begins to operate and pushes β_k to more negative values; the random walk component dominates in the long-run, and so the slopes return to zero at long horizons ($\beta_k \to 0$ as $k \to 0$).

Fig. 2 plots the estimated slope coefficient β_k against the horizon periods k. The U-shaped pattern indicates the existence of both temporary and random walk components. For $k = 60$ (i.e. over a period of 5 years) the regression slopes are -0.36. We can infer that roughly 70% of the variance of a 5-year change in the US nominal short term interest rate is due to the stationary component of the series.[8]

Fig. 1: Variance Ratio Statistic

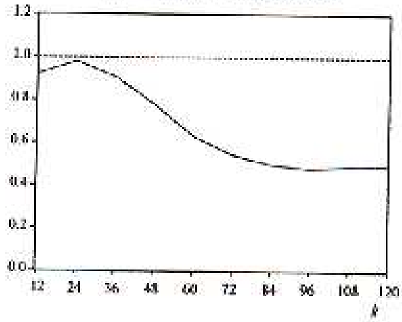
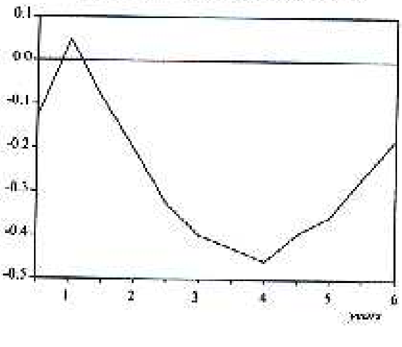


Fig. 2: Estimated Slope Coefficients



4 Conclusion

Most equilibrium models of the term structure assume dynamics of the state variable, i.e. the instantaneous interest rate, that are not congruent with the data. Given the importance of the choice of the specific interest rate process for yield curve modelling, valuation of interest rate sensitive securities and general risk management, we develop a new continuous time term structure model that replicates empirically observed premises.

We take the view that the dynamics of the instantaneous interest rate process are given by the combined effect of a (stationary) mean reverting

[7] See Fama and French (1988), and Huizinga (1987).

[8] It can be shown that, for large k, the ratio of the k-period change in the stationary component to the k-period actual change in the time series is equal to $-2\beta_k$.

component and a (nonstationary) martingale component. The principal result of the paper is a closed-form solution of the yield curve when the state variable is given by a mix of autoregressive and random walk processes.

Such a modelling approach is justified in view of our empirical results. Application of the variance ratio statistic and regression analysis shows that the US short term nominal interest rate is a unit root process with a significant mean reverting component.

References

[1] Arnold, Ludwig, *Stochastic Differential Equations: Theory and Applications* Wiley, New York, 1974.

[2] Beveridge, Stephen, and Nelson, Charles R., A new approach to decomposition of economic time series into permanent and transitory components with particular attention to measurement of the business cycle, *Journal of Monetary Economics*, Vol.7, 1981, pp. 151-174.

[3] Brennan, Michael J., and Eduardo S. Schwartz, Savings bonds, detractable bonds, and callable bonds, *Journal of Financial Economics*, Vol.3, 1977, pp. 135-155.

[4] Brennan, Michael J., and Eduardo S. Schwartz, An equilibrium model of bond pricing and a test of market efficiency, *Journal of Financial and Quantitative Analysis*, Vol.17, 1982, pp. 301-329.

[5] Brennan, Michael J., and Eduardo S. Schwartz, Alternative models for valuing debt options, *Finance*, Vol.4, 1983, pp. 119-138.

[6] Campbell, John Y., and Gregory N. Mankiw, Are output fluctuations transitory?, *Quarterly Journal of Economics*, Vol.102, 1987, pp. 857-880.

[7] Cochrane, John H., How big is the random walk in GNP?, *Journal of Political Economy*, Vol.91, 1988, pp. 893-920.

[8] Cox, John C., Jonathan E. Ingersoll, and Steven A. Ross, A theory of the term structure of interest rates, *Econometrica*, Vol.53, 1985, pp. 385-407.

[9] Dothan, Michael U., On the Term Structure of Interest Rates, *Journal of Financial Economics*, Vol.6, 1978, pp. 59-69.

[10] Fama, Eugene F., and Kenneth R. French, Permanent and temporary components of stock prices, *Journal of Political Economy*, Vol.96, 1988, pp. 246-273.

[11] Huizinga, John , An empirical investigation of the long-run behavior of real exchange rates, *Carnegie-Rochester Conference Series on Public Policy*, Vol.27, 1987, pp. 149-214.

[12] Longstaff, Francis A., and Eduardo S. Schwartz, Interest rate volatility and the term structure: A two factor general equilibrium model, *Journal of Finance*, Vol.47, 1992, pp. 1259-1282.

[13] Merton, Robert C., Theory of rational option pricing, *Bell Journal of Economics and Management Science*, Vol.4, 1973, pp. 114-183.

[14] Merton, Robert C., On the term structure of interest rates and other things, Unpublished Note, MIT, 1974.

[15] Richard, Scott F., An arbitrage model of the term structure of interest rates, *Journal of Financial Economics*, Vol.6, 1978, pp. 33-57.

[16] Vasicek, O., An equilibrium characterization of the term structure, *Journal of Financial Economics*, Vol.5, 1977, pp. 177-188.