The Statistical Properties of Long-memory ACD Models

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Abstract: - This paper examines some of the statistical properties of the long-memory Autoregressive Conditional Duration (ACD) specification. To allow for non-monotonic hazard functions we use the generalized F distribution. Conditions for the existence of the first two moments are established. We also provide analytical expressions of the autocorrelation function of the durations.

Key-words: - Long-memory ACD models, Generalized F distribution, Unconditional moments, Hypergeometric Functions.

1 Introduction

Engle and Russell (1998) proposed a new econometric framework for the modelling of intertemporally correlated event arrival times, termed the Autoregressive Conditional Duration (ACD) model. A feature of Engle and Russell's linear ACD model with exponential or Weibull errors is that the implied conditional hazard functions are restricted to being either constant, increasing or decreasing. Zhang et al. (2001) (among others) questioned whether this assumption is an adequate one. As an alternative to the Weibull distribution used in the original ACD model Hautsch (2001) utilized the generalized F distribution. Recently, several extensions to Engle and Russell's (1998) basic model have been proposed. Bauwens et al. (2000) review several duration models that have been proposed in the literature. Moreover, many authors modelled conditional heteroscedasticity in equidistant financial time series using long-memory models (e.g. Giraitis et al., 2002). To capture the long range time dependence in intertrade durations Jasiak (1998) proposed the fractionally integrated ACD (FIACD) model.

The objective of this paper is to investigate some of the statistical properties of an alternative ACD model that shed light on the dynamics of the transaction arrival process: the long-memory ACD (LMACD) model, which captures the long-term dependencies in the duration series and, in contrast to the FIACD model, is covariance stationary. In addition, since Engle and Russell (1998) find that the Weibull distribution suffers from remaining excess dispersion we use an alternative distribution which includes the Weibull as a special case: the generalized F (GF). For the aforementioned model we provide existence conditions of the second moments of the durations. We also derive analytical expressions for the autocorrelation function (ACF) of the durations.[1] To facilitate model identification, the results for the ACF of the durations can be applied so that properties of the observed data can be compared with the theoretical properties of the models. The significance of our results extends to the development of misspecification tests and estimation.

The rest of the paper is organized as follows. Section 2 briefly reviews the ACD model of Engle and Russell (1998) and considers the properties of

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[1] Karanasos et al. (2003) examined the dependence structure of long-memory GARCH processes.

the generalized F distribution. Section 3 provides a detailed description of the long-memory ACD model and derives the ACF of the durations for this model. Section 4 concludes the paper.

2 The ACD Model

Consider a stochastic point process that is simply a sequence of arrival times $\{t_0, t_1, \ldots, t_N\}$ with $0 \equiv t_0 < t_1 < \cdots < t_N$. Duration (x_i) is the time elapsed between two consecutive arrival times, i.e. $x_i \equiv t_i - t_{i-1}$.

The ACD model of Engle and Russell (1998) specifics the observed duration as a mixing process

$$x_i = \psi_i \varepsilon_i \quad (i \in \mathbb{Z}), \tag{1}$$

where $\{\varepsilon_i\}$ is a sequence of independent and identically distributed random variables with density $\xi(\varepsilon_i;\phi) \equiv \xi(\varepsilon_i)$ (where ϕ is vector of parameters) and mean equal to one, and ψ_i denotes the conditional expectation of the ith duration. That is

$$\psi_i \equiv \psi_i(x_{i-1}, \ldots, x_1; \theta) = E(x_i | I_{i-1}),$$

where I_{i-1} denotes the conditioning information set generated by the durations preceding x_i and θ is a vector of parameters. The flexibility of the model (1) lies in the rich host of candidates for the specification of the dynamic structure of ψ_i as well as the conditional density ξ.

Following Engle and Russell (1998), we express the conditional density of x_i as

$$g(x_i, \psi_i; \phi) \equiv g(x_i) = \frac{1}{\psi_i} \xi\left(\frac{x_i}{\psi_i}\right). \tag{2}$$

The specification in (2) can be generalized in many ways. In what follows we examine a four-parameter general distribution that includes as special or limiting cases many distributions considered in econometrics and finance. Expressions are reported that facilitate analysis of hazard functions, other distributional characteristics and parameter estimation.

2.1 The generalized F distribution

In this section, we examine the GF distribution. It is a particularly useful family of distributions which includes among others the Burr type 12, the Lomax, the Fisk, and the folded t. Its density is given by

$$\xi(\varepsilon_i) = \frac{a \varepsilon_i^{ap-1} q^q \varphi^{aq}}{B(p,q)(q\varphi^a + \varepsilon_i^a)^{p+q}} \; (\varepsilon_i \geq 0, \; i = 1, .., N), \tag{3}$$

where $B(\cdot)$ is the beta function, $a > 0$, and the parameter φ is merely a scale parameter. Further, if the φ coefficient is

$$\varphi = \frac{B(p,q)}{q^{\frac{1}{a}} B\left(p + \frac{1}{a}, q - \frac{1}{a}\right)},$$

then ε_i has mean equal to one. The GF distribution has integer moments of order up to 'μ' where $-p < \frac{\mu}{a} < q$ (see McDonald and Richards, 1987a).

The conditional density of x_i can be written as

$$g(x_i) = \frac{a x_i^{ap-1} q^q (\varphi\psi_i)^{aq}}{B(p,q)[q(\varphi\psi_i)^a + x_i^a]^{p+q}},$$

Using the above expression we can write the log-likelihood function of the observations x_i, $i = 1, \ldots, N$ as

$$\mathcal{L} \equiv \sum_{i=1}^{N} \ln[g(x_i)] = c + \sum_{i=1}^{N} \{(ap-1)\ln(x_i) + aq\ln(\psi$$
$$-(p+q)\ln\left[q(\varphi\psi_i)^a + x_i^a\right]\},$$

where

$$c \equiv N\{\ln(a) + q\ln(q) + aq\ln(\varphi) - \ln[B(p,q)]\}.$$

3 The long-memory ACD model

The ACD model is closely related to the GARCH model and shares some of its features. Just as the simple GARCH model is often a good starting point, the simplest version of the ACD model seems like a natural starting point. However, as there are many alternative volatility models, there is a rich host of candidates for the dynamic specification of the conditional duration. Such specification includes models analogous to long-memory, and many other GARCH models as possibilities.

The standard ACD model accounts for short serial dependence in conditional durations and thus compels the pattern of the autocorrelation function to decay exponentially. In empirical applications of ACD models to high frequency intertrade durations the estimated coefficients on lagged variables sum up nearly to one. Such evidence indicates a potential misspecification that arises when an exponentially declining shape is fitted to a process showing an hyperbolic rate of decay. This would suggest that a more flexible structure allowing for longer term dependencies might improve the fit. In this respect, the motivation for using the long-memory ACD models is to capture the long-term dependencies in the duration series.

In the long-memory GARCH (LMGARCH) model of Robinson and Henry (1999) the conditional variance of the process implies a slow hyperbolic rate of decay for the influence of the squared errors. Analogously to the LMGARCH process for the volatility, the long-memory ACD(n,d,m) [LMACD(n,d,m)] model for the duration is defined by

$$x_i = \omega + \frac{C(L)}{B(L)(1-L)^d}\eta_i, \tag{4}$$

for some $\omega, d \in (0,\infty)$ with

$$B(L) \equiv 1 - \sum_{j=1}^{n} \beta_j L^j = \prod_{j=1}^{n}(1-\lambda_j L),$$
$$C(L) \equiv 1 - \sum_{l=1}^{m} c_l L^l,$$

where $\eta_i \equiv x_i - \psi_i$ is a martingale difference sequence by construction, λ_j is the reciprocal of the jth root of $B(L)$, and d is the fractional differencing parameter. Further, we assume that all the roots of $C(L)$ and $B(L)$ lie outside the unit circle.

Lemma 1 *The duration $\{x_i\}$ can be written as an infinite sum of lagged values of η_i*

$$x_i = \omega + \sum_{j=0}^{\infty} \omega_j \eta_{i-j}, \tag{5a}$$

where

$$\omega_j \equiv \sum_{r=1}^{n} \lambda_r^+ \sum_{l=0}^{j} \binom{-d}{j-l} \pi_{rl}(-1)^{j-l}, \tag{5b}$$

with

$$\pi_{rl} \equiv \sum_{j=0}^{\min\{l,m\}} \lambda_r^{l-j}(-c_j) \quad (c_0 \equiv -1),$$
$$\lambda_r^+ \equiv \frac{\lambda_r^{n-1}}{\prod_{l=1, l\neq r}^{n}(\lambda_r - \lambda_l)}.$$

Proof. See Appendix.

Moreover, ψ_i can be expressed as an infinite distributed lag of x_i terms:

$$\psi_i \equiv \left[1 - \frac{(1-L)^d B(L)}{C(L)}\right] x_i = \sum_{j=1}^{\infty} \phi_j x_{i-j}, \tag{6}$$

with $\phi_1 > 0$, $\phi_j \geq 0$ $(j \geq 2)$ [2].

Jasiak (1998) presented empirical evidence for the presence of long memory, and proposed a fractionally integrated model for high frequency duration data. She applied the following specification to intertrade durations for the IBM and Alcatel stocks

$$B(L)(1-L)^d x_i = \omega + C(L)\eta_i. \tag{7}$$

By analogy to the fractionally integrated GARCH (FIGARCH) model (the FIGARCH model was introduced by Baillie et al., 1996) this process is called fractionally integrated ACD (FIACD) model. For $d > 0$, x_i has an unbounded first moment. However, when $d > 0$ the FIACD process governed by (7) is strictly stationary and ergodic, and the "autocorrelations" $\sum_{j=0}^{\infty} \omega_j \omega_{j+k} / \sum_{j=0}^{\infty} \omega_j^2$ are well defined under $\sum_{j=0}^{\infty} \omega_j^2 < \infty$.

[2] The requirement $0 < \sum_{j=0}^{\infty} \omega_j^2 < \infty$ includes the case $\Omega(L) = \sum_{j=0}^{\infty} \omega_j L^j = C(L)/[B(L)(1-L)^d]$, for $d \in (0, 0.5)$, and finite order polynomials $B(L)$ and $C(L)$ whose zeros are outside the unit circle in the complex plane. In this case the weights ϕ_j satisfy $\sum_{j=0}^{\infty} |\phi_j| < \infty$, $\phi_0 \equiv 1$. Under $\max_i \mathrm{E}(x_i^2) < \infty$ it follows that $\mathrm{E}(\eta_i^2) < \infty$ (see theorem 1 below), so that the innovations in (4) are square integrable martingale differences and x_i is well defined as a covariance stationary process and its autocorrelations $\mathrm{Corr}\,(x_i, x_{i-k}) \equiv \rho_k(x_i) = \sum_{j=0}^{\infty} \omega_j \omega_{j+k} / \sum_{j=0}^{\infty} \omega_j^2$ $(k \in \mathbb{N})$ can exhibit the usual long memory structure implied by $C(L)/[B(L)(1-L)^d]$. Even if $\max_i \mathrm{E}(x_i^2) < \infty$ does not hold, the "autocorrelations" $\sum_{j=0}^{\infty} \omega_j \omega_{j+k} / \sum_{j=0}^{\infty} \omega_j^2$ are well defined under $0 < \sum_{j=0}^{\infty} \omega_j^2 < \infty$.

3.1 Autocorrelation function of the LMACD model

Several previous articles dealing with financial market data-e.g. Dacorogna et al. (1993)- have commented on the behavior of the autocorrelation function of power transformed absolute returns, and the desirability of having a model which comes close to replicating certain stylized facts in the data (abstracted from Baillie and Chung, 2001). In this respect, one can apply the results in this section to check whether the long-memory ACD model can effectively replicate the observed pattern of autocorrelations of the durations.

Another potential motivation for the derivation of the results in this section is that the autocorrelations of the durations in (1) and (4) can be used to estimate the ACD parameters in (4). The approach is to use the minimum distance estimator (MDE), which estimates the parameters by minimizing the Mahalanobis generalized distance of a vector of sample autocorrelations from the corresponding population autocorrelations (see Baillie and Chung, 2001).

In this section we consider the LMACD(n, d, m) process defined by (1) and (4) with $d \in (0, 0.5)$ and the additional restriction that the roots of $B(z) = 0$ are simple.

Lemma 2 *The condition for the existence of the second moment of the duration is*

$$\left[1 - \frac{1}{\mathsf{E}(\epsilon_i^2)}\right] \left(\sum_{j=0}^{\infty} \omega_j^2\right) < 1,$$

where ω_j is defined by (5b).

Proof. See Appendix.

In the following theorem we establish a representation for the autocorrelation function of the durations of the LMACD(n, d, m) process, with $d \in (0, \frac{1}{2})$. Let F be the Gaussian hypergeometric function defined by $F(a, b; c; z) \equiv \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}$, where $(b)_j \equiv \prod_{i=0}^{j-1} (b + i)$ is Pochhammer's shifted factorial.

Theorem 1 *The autocorrelation function of $\{x_i\}$ can be expressed as*

$$\rho_k(x_i) = \frac{\sum_{j=0}^{\infty} \omega_j \omega_{j+k}}{\sum_{j=0}^{\infty} \omega_j^2} = \frac{z_k}{z_0}, \quad (k \in \mathbb{N}), \qquad (8)$$

where

$$z_k \equiv \sum_{j=1}^{n} \sum_{l=0}^{m} \Psi_l \overline{\lambda}_j C(d, k, l, \lambda_j) \quad (k \geq 0),$$

with

$$\Psi_l \equiv \sum_{r=0}^{m-l} c_r c_{r+l} \ (c_0 \equiv -1),$$

$$\overline{\lambda}_j \equiv \frac{\lambda_j^+}{\prod_{r=1}^{n} (1 - \lambda_j \lambda_r)},$$

$$\begin{aligned} C(d, k, l, \lambda_j) \equiv\ & \frac{\Gamma(d + k + l)}{\Gamma(1 - d + k + l)} \times \\ & F(d + k + l, 1; 1 - d + k + l; \lambda_j) + \\ & 1_l \frac{\Gamma(d + k - l)}{\Gamma(1 - d + k - l)} \times \\ & F(d - k + l, 1; 1 - d - k + l; \lambda_j) + \\ & \lambda_j [1_l \frac{\Gamma(d + k + 1 - l)}{\Gamma(2 - d + k - l)} \\ & \times F(d + k + 1 - l, 1; 2 - d + k - l; \lambda_j) + \\ & \frac{\Gamma(d + k - 1 + l)}{\Gamma(k - d + l)} \times \\ & F(d - k + 1 - l, 1; 2 - d - k; \lambda_j)], \end{aligned}$$

and

$$1_l \equiv \begin{cases} 0, & \text{if } l = 0, \\ 1, & \text{if } l \neq 0 \end{cases},$$

if and only if $\left[1 - \frac{1}{\mathsf{E}(\epsilon_i^2)}\right] z_0 \leq 1$.

Proof. See Appendix.

To illustrate the general result we consider the LMACD(1,d,1) process. In this case $1 - C(L) = c_1 L \equiv cL$ and $1 - B(L) = \beta_1 L \equiv \beta L$.

Corollary 1 *For the LMACD(1,d,1) model, with $|c|, |\beta| < 1$ and $d \in (0, \frac{1}{2})$, the autocorrelation function of $\{x_i\}$ is given by*

$$\rho_k(x_i) = \frac{z_k}{z_0} \quad (k \in \mathbb{N}), \qquad (9)$$

where

$$\begin{aligned}
z_k \equiv\ & \frac{\Gamma(1-2d)}{(1-\beta^2)\Gamma(d)\Gamma(1-d)} \times \\
& \{\frac{\Gamma(d+k)}{\Gamma(1-d+k)}[(1+c^2-\beta c) \times \\
& F(d+k,1;1-d+k;\beta) - \\
& c\beta F(d-k,1;1-d-k;\beta)] + \\
& \frac{\Gamma(d+k-1)}{\Gamma(k-d)}[\beta(1+c^2)-c] \times \\
& F(d-k+1,1;2-d-k;\beta) - \frac{\Gamma(d+k+1)}{\Gamma(2-d+k)} \times \\
& cF(d+k+1,1;2-d+k;\beta)\} \quad (k \geq 0),
\end{aligned}$$

if and only if $\left[1-\frac{1}{\mathsf{E}(\varepsilon_i^2)}\right] z_0 \leq 1$.

Proof. The proof follows from lemma 2 and theorem 1, by setting $C(L) \equiv 1-cL$ and $B(L) \equiv 1-\beta L$. ■

Fig. 1 plots the theoretical ACF of the duration of the above process (for various values of the three parameters c, β and d).

Remark 1. As expected, the autocorrelation function of the LMACD(1,1) process decays at a very slow rate. When $c = 0.6$, $\beta = 0$ and $d = .45$, for instance, the duration still has an autocorrelation coefficient around 0.5 at lag 1,000.

4 Conclusion

This paper has provided a detailed description of the long-memory ACD model. We also investigated the properties of the generalized F distribution which allows for non-monotonic hazard functions. For all ACD specifications we derived analytical expressions of the autocorrelation function of the durations. Conditions for the existence of the first two moments of the durations were also established. The derivation of the autocorrelations of the durations and their comparison with the corresponding sample equivalents will help the investigator (a) to decide which is the most appropriate method of estimation (e.g. maximum likelihood estimation (MLE), minimum distant estimator (MDE)) for a specific model, (b) to chose, for a given estimation technique, the model that best replicates certain stylized facts of the data and, (c) in conjunction with the various model selection criteria, to identify the optimal order of the chosen specification.

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Fig. 1: Autocorrelation function of x_i. FIACD(1,1) model.

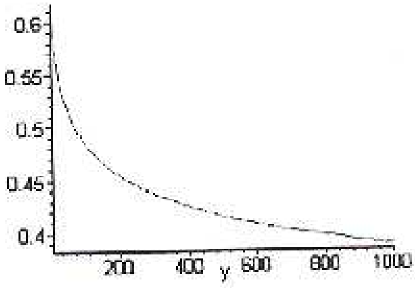

Fig. 1a: $c = 0.2$, $\beta = 0.3$, $d = 0.45$.

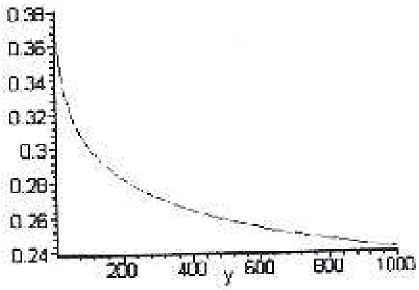

Fig. 1b: $c = 0.2$, $\beta = 0.6$, $d = 0.45$.

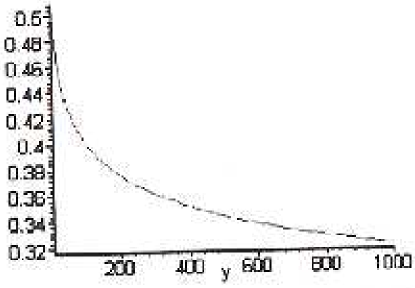

Fig. 1c: $c = 0.4$, $\beta = 0.6$, $d = 0.45$.

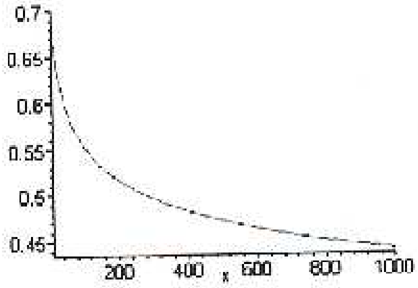

Fig. 1d: $c = 0.2$, $\beta = 0$, $d = 0.45$.

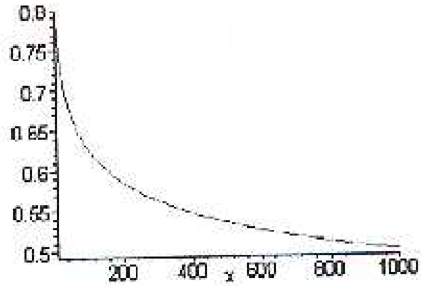

Fig. 1e: $c = 0.6$, $\beta = 0$, $d = 0.45$.

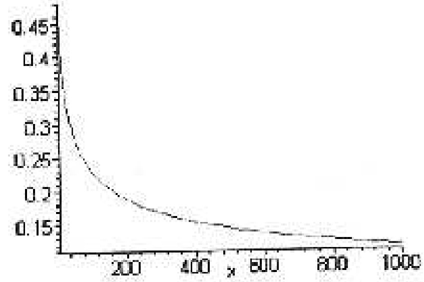

Fig. 1f: $c = 0.6$, $\beta = 0$, $d = 0.35$.

Appendix

Proof. [Lemma 1] From (4) we have that

$$x_i = \omega + (1-L)^{-d}\left[\prod_{j=1}^{n}(1-\lambda_j L)\right]^{-1} C(L)\eta_i,$$

where the operator $(1-\gamma L)^{-d}$ is defined as

$$(1-\gamma L)^{-d} = \sum_{j=0}^{\infty} \binom{-d}{j} (-\gamma)^j L^j.$$

Hence, on account of

$$\prod_{j=1}^{n} (1-\lambda_j L) = \sum_{j=1}^{n} \frac{\lambda_j^{n-1}}{\prod_{l=1, l\neq j}^{n} (\lambda_j - \lambda_l)(1-\lambda_j L)},$$

and

$$(1-\lambda_j L)^{-1} C(L) = \sum_{f=0}^{\min\{l,m\}} \sum_{l=0}^{\infty} \lambda_j^{l-f} (-c_f) L^l,$$

we obtain (5a). ■

Proof. [Lemma 2] Rewriting equation (5a) we have

$$x_i = \omega + \sum_{j=0}^{i} \omega_j \eta_{i-j}, \qquad (10)$$

where ω_j is defined in (5b).

From (10) it follows that

$$\mathsf{Var}(x_i) = \left(\sum_{j=0}^{\infty} \omega_j^2 \right) \mathsf{E}(\eta_i^2). \qquad (11)$$

Using the fact that

$$\mathsf{E}(\eta_i^2) = \mathsf{E}(x_i^2) \left[1 - \frac{1}{\mathsf{E}(\varepsilon_i^2)} \right],$$

we obtain the second moment of the duration

$$\mathsf{E}\left(x_i^2\right) = \frac{[\mathsf{E}(x_i)]^2}{1 - \left[1 - \frac{1}{\mathsf{E}(\varepsilon_i^2)}\right] \left(\sum_{j=0}^{\infty} \omega_j^2 \right)}.$$

■

Proof. [Theorem 1] Applying (5b) yields

$$\rho_k(x_i) = \frac{\sum_{j=0}^{\infty} \omega_j \omega_{j+k}}{\sum_{j=0}^{\infty} \omega_j^2} = \frac{\sum_{r=0}^{\infty} \sum_{j=-\infty}^{\infty} \phi_{r,k-j} v_j}{\sum_{r=0}^{\infty} \sum_{j=-\infty}^{\infty} \phi_{r,-j} v_j},$$

where

$$\phi_{r,k-j} \equiv \binom{-d}{|k-j|+r} \binom{-d}{r} (-1)^{|k-j|},$$

$$v_j \equiv \sum_{l=1}^{n} \sum_{s=1}^{n} \sum_{f=0}^{\infty} \lambda_l^+ \lambda_s^+ \pi_{lf} \pi_{s,f+|j|},$$

with

$$\pi_{lf} \equiv \sum_{r=0}^{\min\{f,m\}} \lambda_l^{f-r} (-c_r),$$

$$\lambda_l^+ \equiv \frac{\lambda_l^{n-1}}{\prod_{r=1, r\neq l}^{n} (\lambda_l - \lambda_r)}.$$

But since

$$\sum_{r=0}^{\infty} \phi_{r,k-j} = \frac{\Gamma(1-2d)\Gamma(d+|k-j|)}{\Gamma(d)\Gamma(1-d)\Gamma(1-d+|k-j|)},$$

and

$$\sum_{r=1}^{n} \frac{\lambda_l^+ \lambda_r^+}{1-\lambda_l \lambda_r} \equiv \overline{\lambda}_l,$$

it follows that

$$\rho_k(x_i) = \frac{\sum_{j=-\infty}^{\infty} \frac{\Gamma(d+|k-j|)}{\Gamma(1-d+|k-j|)} \sum_{r=1}^{n} \sum_{l=0}^{m} \Psi_l \overline{\lambda}_r (1_l \lambda_r^{||j|-l|} + \lambda_r^{|j|+l})}{\sum_{j=-\infty}^{\infty} \frac{\Gamma(d+|j|)}{\Gamma(1-d+|j|)} \sum_{r=1}^{n} \sum_{l=0}^{m} \Psi_l \overline{\lambda}_r (1_l \lambda_r^{||j|-l|} + \lambda_r^{|j|+l})} -$$
$$\left\{ \sum_{l=0}^{m} \sum_{j=0}^{\infty} \Psi_l \sum_{r=1}^{n} \overline{\lambda}_r \left[\frac{1_l \Gamma(d+|k-l-j|)}{\Gamma(1-d+|k-l-j|)} + \frac{\Gamma(d+k+l+j)}{\Gamma(1-d+k+l+j)} + \right.\right.$$
$$\left.\left. \left(\frac{\Gamma(d+|k-1+l-j|)}{\Gamma(1-d+|k-1+l-j|)} + \frac{1_l \Gamma(d+|k+1-l+j|)}{\Gamma(1-d+|k+1-l+j|)} \right) \lambda_r \right] \lambda_r^j \right\} \times$$
$$\left\{ \sum_{l=0}^{m} \sum_{j=0}^{\infty} \Psi_l \sum_{r=1}^{n} \overline{\lambda}_r \left[\frac{1_l \Gamma(d+|-l-j|)}{\Gamma(1-d+|-l-j|)} + \frac{\Gamma(d+l+j)}{\Gamma(1-d+l+j)} + \right.\right.$$
$$\left.\left. \left(\frac{\Gamma(d+|-1+l-j|)}{\Gamma(1-d+|-1+l-j|)} + \frac{1_l \Gamma(d+|1-l+j|)}{\Gamma(1-d+|1-l+j|)} \right) \lambda_r \right] \lambda_r^j \right\}^{-1},$$

where

$$\Psi_l \equiv \sum_{r=0}^{m-l} c_r c_{r+l} \;\; (c_0 \equiv -1), \;\; \overline{\lambda}_l \equiv \frac{\lambda_l^+}{\prod_{r=1}^{n} (1-\lambda_l \lambda_r)}.$$

Hence, upon observing that

$$\sum_{j=0}^{\infty} \frac{\Gamma(d+k+l+j)}{\Gamma(1-d+k+l+j)} \lambda_r^j =$$
$$\frac{\Gamma(d+k+l)}{\Gamma(1-d+k+l)} F(d+k+l, 1; 1-d+k+l; \lambda_r),$$

(8) is obtained. ■