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The Statistical Properties of Exponential ACD Models

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Abstract

This paper examines some of the statistical properties of exponential ACD models. To allow for non-monotonic hazard functions we use either the generalized Gamma or the generalized F distributions. Conditions for the existence of the first two moments of the durations are established. We also provide analytical expressions of the autocorrelation function of the durations.

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1 Introduction

Engle and Russell (1998) propose a new econometric framework for the modelling of intertemporally correlated event arrival times, termed the Autoregressive Conditional Duration (ACD) model. A feature of Engle and Russell's linear ACD specification with exponential or Weibull errors is that the implied conditional hazard functions (CHF) are restricted to being either constant, increasing or decreasing. Zhang, Russell and Tsay (2001), Hamilton and Jorda (2002) and Bauwens and Veredas (2004) questioned whether this assumption is an adequate one. As an alternative to the Weibull distribution used in the original ACD model, Lunde (1999) employs a formulation based on the generalized Gamma (GG) distribution, while Grammig and Maurer (2000) and Hautsch (2001) utilize the Burr and generalized F (GF) distributions respectively.

The last ten years or so, several extensions to Engle and Russell's (1998) basic process have been proposed. Bauwens, Giot, Grammig and Veredas (2004) and Pacurar (2008) review several duration models that have been proposed in the literature.[1] Lunde (1999) and Bauwens and Giot (2000, 2001a) model the effect of recent durations on the conditional mean with a logarithmic transformation. We call this model exponential ACD (EACD) because of its resemblance to Nelson's EGARCH model (Nelson, 1991).[2] Giot (2001) uses it to obtain a direct estimate of the intraday volatility for the IBM stock traded on the NYSE,. Dufour and Engle (2000) suggest a version of it which allows for an asymmetric response to innovation shocks. The exponential formulation avoids some of the parameter restrictions postulated by the original ACD specification.

The objective of this paper is to investigate some of the statistical properties of two versions of the EACD specification, which allow us to obtain analytically the unconditional moments implied by the model. In addition, since Engle and Russell (1998) find that the Weibull distribution suffers from remaining excess dispersion we use two alternative distributions which include the Weibull as a special case: the GF and the GG.[3] For the aforementioned models we provide existence conditions of the second moments of the durations. We also derive analytical expressions for the autocorrelation function (ACF) of the durations.[4] Bauwens and Giot (2001a) and Bauwens, Galli and

[1] Recent contributions include Grammig and Wellner (2002), Bauwens and Giot (2003), Li and Wu (2003), Feng, Jiang and Song (2004), Fernandes (2004), Ghysels, Gouriéroux and Jasiak (2004), Focardi and Fabozzi (2005), Giot (2005), Fernandes, Medeiros and Veiga (2006) and Meitz and Teräsvirta (2006).

[2] This model is termed logarithmic ACD(LACD) by Bauwens and Giot (2000).

[3] Details on the properties of this distribution can also be found in Bauwens and Giot (2001b).

[4] He, Teräsvirta and Malmsten (2002), Demos (2002) and Karanasos and Kim (2003) studied the moment structure of exponential GARCH models. Karanasos, Psaradakis and Sola (2003) examined

Giot (2008) also provide analytical expressions for the unconditional moments and ACF for the models belonging to the EACD class as defined in Bauwens and Giot (2000). We should also mention that Carrasco and Chen (2002) provide sufficient conditions to ensure β-mixing and finite higher order moments for various linear and nonlinear GARCH, stochastic volatility, and ACD models. To facilitate model identification, the results for the ACF of the durations can be applied so that properties of the observed data can be compared with the theoretical properties of the models. The significance of our results extends to the development of misspecification tests and estimation.

The rest of the paper is organized as follows. Section 2 briefly reviews the ACD model of Engle and Russell (1998) and considers the properties of the GG and GF distributions. Section 3 provides a detailed description of the EACD model. In addition, it derives the ACF of the durations for its various versions. Section 4 concludes the paper. The proofs are relegated to the Appendix.

2 The ACD Model

Consider a stochastic point process that is simply a sequence of arrival times $\{t_0, t_1,...,$ $t_N, \ldots\}$ with $0 = t_0 < t_1 < \cdots < t_N < \cdots$. Duration (x_i) is the time elapsed between two consecutive arrival times, i.e. $x_i = t_i - t_{i-1}$, $i = 1, 2, \ldots$.

The ACD model of Engle and Russell (1998) specifies the observed duration as a mixing process

$$x_i = \psi_i \varepsilon_i, \tag{1}$$

where $\{\varepsilon_i\}$, $\varepsilon_i \geq 0 \; \forall \; i$, is a sequence of independent and identically distributed random variables with density $\xi(\epsilon_i; \phi) \triangleq \xi(\varepsilon_i)$ (where ϕ is a vector of parameters) and mean equal to one, and ψ_i denotes the conditional expectation of the ith duration. That is

$$\psi_i \triangleq \psi_i(x_{i-1}, \ldots, x_1; \theta) = E(x_i | I_{i-1}),$$

where I_{i-1} denotes the conditioning information set generated by the durations preceding x_i and θ is a vector of parameters.

The flexibility of the model (1) lies in the rich host of candidates for the specification of the dynamic structure of ψ_i as well as the conditional density ξ.

the dependence structure of long-memory GARCH processes. To capture the long range time dependence in intertrade durations Jasiak (1998) proposes the fractionally integrated ACD (FIACD) model. A similar structure that captures the long-term dependencies in the duration series is the long-memory ACD (LMACD) introduced by Karanasos (2001, 2004).

Following Engle and Russell (1998), we express the conditional density of x_i as

$$g(x_i, \psi_i; \phi) \triangleq g(x_i) = \frac{1}{\psi_i}\xi\left(\frac{x_i}{\psi_i}\right). \tag{2}$$

The above specification can be generalized in many ways. The hazard functions can be given many parametric shapes. ACD formulations with exponential or Weibull errors imply that the CHF must either increase, decrease or stay constant during a time-spell. Grammig and Maurer (2000) investigate whether the restrictions concerning the CHF can imperil the successful application of ACD models. In a simulation study they show that the quasi maximum likelihood estimators of the basic ACD process tend to be biased and inefficient when the true data generating process required non-monotonic hazard functions. In addition, bias and inefficiency also affected the estimators of the parameters that were needed to predict expected durations. This entails severe consequences for the class of GARCH processes for irregularly spaced data, recently introduced by Engle (2000) and Ghysels and Jasiak (1998), in which ACD models are employed to predict conditional expected durations that enter the conditional heteroskedasticity equation in the form of explanatory variables (abstracted from Grammig and Maurer, 2000).

Lunde (1999) and Grammig and Maurer(2000) -who use ACD models with GG and Burr errors respectively- find that for price duration processes of NYSE traded stocks non-monotonic shapes of the hazard functions are indicated. Both studies, using standard likelihood ratio tests, reject the exponential and Weibull ACD specifications in favor of the GG and Burr ACD formulations respectively. Accordingly, in what follows we examine two three/two-parameter general distributions that include as special or limiting cases many distributions considered in econometrics and finance. Expressions are reported that facilitate analysis of hazard functions, other distributional characteristics and parameter estimation.

2.1 The Generalized F Distribution

First, we examine the GF distribution. This is a particularly useful family of distributions, which includes among others the Burr type 12, the Lomax, the Fisk, and the folded t. Its density is given by

$$\xi(\varepsilon_i) = \frac{a\varepsilon_i^{ap-1}q^q\varphi^{aq}}{B(p,q)(q\varphi^a + \varepsilon_i^a)^{p+q}}, \tag{3}$$

where $B(\cdot,\cdot)$ is the beta function, $a > 0$, and the parameter φ is merely a scale parameter. Further, if the φ coefficient is

$$\varphi = \frac{B(p,q)}{q^{\frac{1}{a}} B\left(p + \frac{1}{a}, q - \frac{1}{a}\right)},$$

then ε_i has mean equal to one. The GF distribution has integer moments of order up to 'μ' where -$p < \frac{\mu}{a} < q$ (see McDonald and Richards, 1987a). In a recent paper Hautsch (2006) uses the GF distribution, which allows for a wide range of possible hazard shapes. For volume durations he found that it provides a significantly better fit than the exponential distribution. The mathematical expression for the CHF of the GF distribution is given in McDonald and Richards (1987b).

The conditional density of x_i can be written as

$$g(x_i) = \frac{a x_i^{ap-1} q^q (\varphi\psi_i)^{aq}}{B(p,q)[q(\varphi\psi_i)^a + x_i^a]^{p+q}}.$$

Using the above expression we can write the log-likelihood function of the observations x_i, $i = 1, \ldots, N$ as

$$\mathcal{L} = \sum_{i=1}^{N} \ln[g(x_i)] = c + \sum_{i=1}^{N} \{(ap-1)\ln(x_i) + aq\ln(\psi_i) - (p+q)\ln\left[q(\varphi\psi_i)^a + x_i^a\right]\},$$

where

$$c = N\{\ln(a) + q\ln(q) + aq\ln(\varphi) - \ln[B(p,q)]\}.$$

2.2 The Generalized Gamma Distribution

In this subsection we examine the GG distribution. This is a particularly useful family of distributions, which includes among others the Weibull, the generalized Rayleigh, the Rayleigh, the exponential, and the half normal. Its density is given by

$$\xi(\varepsilon_i) = \frac{a\varepsilon_i^{ap-1} e^{-\left(\frac{\varepsilon_i}{\varphi}\right)^a}}{\varphi^{ap}\Gamma(p)}, \tag{4}$$

where $\Gamma(\cdot)$ denotes the gamma function and the parameter φ is merely a scale parameter. In order to normalize the mean to be equal to one, the φ coefficient should be

$$\varphi = \frac{\Gamma(p)}{\Gamma(p + \frac{1}{a})}.$$

The GG distribution has defined moments of order 'μ', where $\frac{\mu}{a}+p>0$. Consequently, for $a>0$, moments of positive integer order are defined. The CHF for the GG function is examined in Glaser (1980). Lunde (1999), using a GG specification, finds that the inverted U-shaped form is strongly supported by the data. As documented by Bauwens and Veredas (2004) the hazard function of several types of financial durations may be increasing for small durations and decreasing for long durations. To account for this stylized fact, Bauwens, Giot, Grammig and Veredas (2004), used the GG distribution, which has two shape parameters and breaks the one-to-one correspondence between the properties of overdispersion (underdispersion) and of decreasing (increasing) hazard.

Moreover, the conditional density of x_i can be written as

$$g(x_i)=\frac{ax_i^{ap-1}e^{-\left(\frac{x_i}{\varphi\psi_i}\right)^a}}{(\varphi\psi_i)^{ap}\Gamma(p)}.$$

The log-likelihood function for the generalised Gamma ACD (GG-ACD) model is

$$\mathcal{L}=c+\sum_{i=1}^{N}\left\{(ap-1)\ln(x_i)-\left(\frac{x_i\Gamma\left(p+\frac{1}{a}\right)}{\psi_i\Gamma(p)}\right)^a-ap\ln(\psi_i)\right\},$$

with

$$c=N\left\{\ln(a)-(1+ap)\ln[\Gamma(p)]+ap\ln\left[\Gamma\left(p+\frac{1}{a}\right)\right]\right\}.$$

Lunde (1999) applies the GG-ACD model to a random sample consisting of seven stocks from the fifty stocks with the highest capitalization value on the NYSE. He finds that the suggested generalization outperformed the formulation employed by Engle and Russell (1998).

3 Exponential ACD Models

The ACD model is closely related to the GARCH specification and shares some of its features. Just as the simple GARCH formulation is often a good starting point, the simplest version of the ACD model seems like a natural starting point. However, as there are many alternative volatility models, there is a rich host of candidates for the dynamic specification of the conditional duration. Such formulations include models analogous to long-memory, exponential and many other GARCH models as possibilities. Fernandes and Grammig (2005) and Hautsch (2006) present a classification of different types of ACD models.

One limitation of the linear ACD formulation results from the non-negativity constraints on the parameters which are imposed to ensure that ψ_i remains non-negative for all i. These constraints imply that an increasing $f(\varepsilon_i)$ in any period increases ψ_{i+j} for all $j \geq 1$, ruling out oscillatory behavior in the ψ_i process. Alternatively, one can use a model in which the logarithm of the conditional duration follows an ACD-like process. This is analogous to Nelson's (1991) EGARCH model for the conditional variance.

Consider the following EACD(n, m) process

$$B(L)\ln(\psi_i) = \omega + C(L)f(\varepsilon_i), \tag{5}$$

with

$$\begin{aligned} B(L) &= 1 - \sum_{j=1}^{n} \beta_j L^j = \prod_{j=1}^{n}(1 - \lambda_j L), \\ C(L) &= \sum_{s=1}^{m} c_s L^s. \end{aligned}$$

Note that λ_j is the reciprocal of the jth root of the autoregressive polynomial $B(L)$. We also assume that β_n and c_m are both not equal to zero.

We will call the specifications in the class where

$$f(\varepsilon_i) = \ln(\varepsilon_i), \tag{6}$$

and the innovations ε_i follow the GG distribution, GG-EACD$_1$. When the conditional distribution of the durations is the GF, these models will be called GF-EACD$_1$.

Furthermore, models in the class where

$$f(\varepsilon_i) = \varepsilon_i^a, \tag{7}$$

and the innovations are drawn from the GG distribution,[5] will be called GG-EACD$_2$.

Bauwens and Giot (2000) propose a version of the EACD$_2$ model, with $f(\varepsilon_i) = \varepsilon_i$ and Weibull errors, under the name logarithmic ACD. Bauwens and Giot (2000) apply this ACD specification to price durations relative to the bid-ask quote process of three securities listed on the NYSE. Lunde (1999), using duration data for seven stocks traded on the NYSE, estimates a version of the GG-EACD$_2$(1,1) model with $f(\varepsilon_i) = \varepsilon_i$. Giot (2000) uses an EACD(1,1) model to compute the Value-at-Risk for three stocks traded

[5] In equation (7) a is one of the parameters of the GG distribution (see equation (4)).

on the NYSE. Bauwens and Giot (2001a), using duration data for several stocks traded on the NYSE, estimate EACD(1,1) models with an exponential distribution for the error term.

3.1 Moments of Exponential ACD Models

Durations between stock market events are often characterized by overdispersion. Another important stylized fact is the shape of the ACF of the durations, which usually decreases slowly from a relatively low positive first-order autocorrelation. It is essential that, for some parameter values, the ACD models can accommodate such stylized facts. In the light of this, it is important (a) to know whether the general shape of the sample autocorrelations is captured by the estimated model and (b) to find the specification for which the estimated theoretical ACF is closest to the ACF of the data and, therefore, has the best fit as far as the ACF is concerned. To gain further insight into how well an ACD formulation fits the data we need to check whether the unconditional moments computed from the analytical formulae are in line with the empirical ones. Therefore, analytical results on the moments and autocorrelations of the durations for the EACD models can indicate whether these specifications provide a better alternative to the standard ACD process.

Assumption 1. The polynomials $B(L)$ and $C(L)$ in equation (5) have no common roots.
Assumption 2. $B(z) = 0$ and $C(z) = 0$ have no roots in the closed disc $\{z : |z| \leq 1\}$.

In what follows we only examine the case where the roots of the autoregressive polynomial $B(L)$ are distinct. In order to simplify the description of our analysis we will introduce the following notation: $\delta(l) = \delta_l$.

Lemma 1 *Under assumptions 1 and 2, the* μth power *of the duration for the EACD model in equations (1) and (5) can be expressed as*

$$x_i^{\mu} = e^{\frac{\mu\omega}{B(1)}} \varepsilon_i^{\mu} \prod_{l=1}^{\infty} [e^{\mu_l f(\varepsilon_{i-l})}] \qquad (\mu \in \mathbb{R}_+), \tag{8}$$

with

$$\mu_l \triangleq \mu(l) = \mu\delta_l = \mu \sum_{j=1}^{n} \zeta_j z_{jl},$$

and

$$\begin{aligned} \zeta_j &= \frac{\lambda_j^{n-1}}{\prod_{f=1, f\neq j}^{n} (\lambda_j - \lambda_f)}, \\ z_{jl} &= \begin{cases} \sum_{r=0}^{l-1} c_{l-r}\lambda_j^r, & \text{if} \quad l \leq m, \\ z_{jm}\lambda_j^{l-m}, & \text{if} \quad l > m \end{cases}, \end{aligned}$$

where c_s, λ_j *are defined in equation* (5).

Lemma 2 *Let assumptions 1 and 2 hold. Suppose further that* $\mathsf{E}(\varepsilon_i^{2\mu})$, $\mathsf{E}(\mathrm{e}^{2\mu_l f(\varepsilon_i)})$, *and* $\mathsf{E}(\varepsilon_i^{\mu}\mathrm{e}^{\mu_l f(\varepsilon_i)})$ *are finite for all* l. *Then the* 2μ*th moment of the duration and the* kth ($k \in \mathbb{N}$) *autocorrelation of the* μ*th power of the duration,* $\rho_k(x_i^{\mu})$, *for the ACD model in equations (1) and (5), have the form*

$$\mathsf{E}(x_i^{2\mu}) = e^{\frac{2\mu\omega}{B(1)}}\mathsf{E}(\varepsilon_i^{2\mu})\prod_{l=1}^{\infty}[\mathsf{E}(e^{2\mu_l f(\varepsilon_{i-l})})], \tag{9}$$

$$\begin{aligned}\rho_k(x_i^{\mu}) = {} & \mathsf{E}(\varepsilon_i^{\mu})\left\{\mathsf{E}[\varepsilon_{i-k}^{\mu}e^{\mu_k f(\varepsilon_{i-k})}]\prod_{r=1}^{k-1}[\mathsf{E}(e^{\mu_r f(\varepsilon_{i-r})})]\prod_{l=1}^{\infty}[\mathsf{E}(e^{(\mu_{k+l}+\mu_l)f(\varepsilon_{i-k-l})})]\right. \\ & \left. - \mathsf{E}(\varepsilon_i^{\mu})\left[\prod_{l=1}^{\infty}[\mathsf{E}(e^{\mu_l f(\varepsilon_{i-l})})]\right]^2\right\}\cdot \\ & \left\{\mathsf{E}(\varepsilon_i^{2\mu})\prod_{l=1}^{\infty}\left[\mathsf{E}(e^{2\mu_l f(\varepsilon_{i-l})})\right] - [\mathsf{E}(\varepsilon_i^{\mu})]^2\left[\prod_{l=1}^{\infty}[\mathsf{E}(e^{\mu_l f(\varepsilon_{i-l})})]\right]^2\right\}^{-1},\end{aligned} \tag{10}$$

where $\prod_{r=1}^{0}[\cdot] = 1$.

Remark 1. For the practical computation of the moments in lemma 2 , the infinite products that appear in equations (9)-(10) can be truncated after a sufficiently large number of terms since μ_l tends to zero. In practice, we found that for first and second moments, truncation after 1000 terms was more than sufficient to get a high accuracy (see also Bauwens and Giot, 2001a and Bauwens, Galli and Giot, 2008).

The 2μth moment and the kth autocorrelation of the duration, for the EACD models, have been independently obtained by Bauwens, Galli and Giot (2008) (see also Bauwens and Giot, 2001a). When considering the EACD(n, m) model in equation (5), theorems 1 and 2 in Bauwens, Galli and Giot (2008) and equations (9)-(10) are equivalent.

Fernandes and Grammig (2006) propose an EACD model with $f(\varepsilon_i) = [|\varepsilon_i - b| + \gamma(\varepsilon_i - b)]^{\upsilon}$, under the name asymmetric Box-Cox ACD (ABC-ACD). This specification provides a flexible functional form that permits the logarithm of the conditional duration to respond in distinct manners to small and large shocks. For the ABC-ACD(1,1) process they report expressions for the 2μth moment and kth autocovariance of the duration which are similar to our equations (9)-(10).

Next, recall that $B(\cdot,\cdot)$ indicates the beta function and p, q, a are the parameters of the GF distribution (see equation (3)).

Theorem 1 *Suppose that assumptions 1 and 2 hold and that both* $\frac{2\mu}{a}, \frac{2\mu_l}{a} \in (-p, q)$. *Then the* 2μ*th moment of the duration and the* kth *autocorrelation of the* μ*th power of the duration, for the GF-EACD*$_1(n, m)$ *model in equations (1), (3) and (5)-(6), are given by*

$$\mathsf{E}(x_i^{2\mu}) = e^{\frac{2\mu\omega}{B(1)}} B_{2\mu} \prod_{l=1}^{\infty} (B_{2\mu_l}), \tag{11}$$

$$\rho_k(x_i^{\mu}) = \frac{B_\mu \left\{ B_{\mu+\mu_k} \prod_{r=1}^{k-1}(B_{\mu_r}) \prod_{l=1}^{\infty}(B_{\mu_{k+l}+\mu_l}) - B_\mu \left[\prod_{l=1}^{\infty}(B_{\mu_l})\right]^2 \right\}}{B_{2\mu} \prod_{l=1}^{\infty}(B_{2\mu_l}) - (B_\mu)^2 \left[\prod_{l=1}^{\infty}(B_{\mu_l})\right]^2}, \tag{12}$$

with

$$B_{\mu_l} \triangleq B(\mu_l) = \frac{[B(p,q)]^{(\mu_l - 1)} B\left(p + \frac{\mu_l}{a}, q - \frac{\mu_l}{a}\right)}{\left[B\left(p + \frac{1}{a}, q - \frac{1}{a}\right)\right]^{\mu_l}},$$

where μ_l *is defined in lemma 1.*

To illustrate the preceding general theory we consider the $\text{EACD}_1(1,1)$ process when the distribution of the innovations is either Lomax or Fisk. These models will be called Lomax EACD_1 (L-EACD_1) and Fisk EACD_1 (F-EACD_1) respectively. In these cases $\mu_l = \mu c \beta^{l-1}$, where $c = c_1$, $\beta = \beta_1$. We have the following corollaries.

Corollary 1a *For the F-EACD*$_1$*(1,1) model, with* $a = 4$ *and* $|c|, |\beta| < 1$, *the* k*th autocorrelation of the duration is given by equation (12) with*

$$B_{\mu_l} = \frac{\Gamma\left(1 + \frac{\mu_l}{4}\right) \Gamma\left(1 - \frac{\mu_l}{4}\right)}{\left[\Gamma\left(\frac{5}{4}\right) \Gamma\left(\frac{3}{4}\right)\right]^{\mu_l}}, \quad \mu_l = c\beta^{l-1}.$$

Proof. The proof follows from theorem 1, by setting $p = q = \mu = 1$, $a = 4$ and $\mu_l = c\beta^{l-1}$. ■

Corollary 1b *For the L-EACD*$_1$*(1,1) model, with* $q = 3$, $-0.50 < c < 1$ $(c \neq 0)$, $|\beta| < 1$ *and* $-0.50 < c\beta$, *the* k*th autocorrelation of the duration is given by equation (12) with*

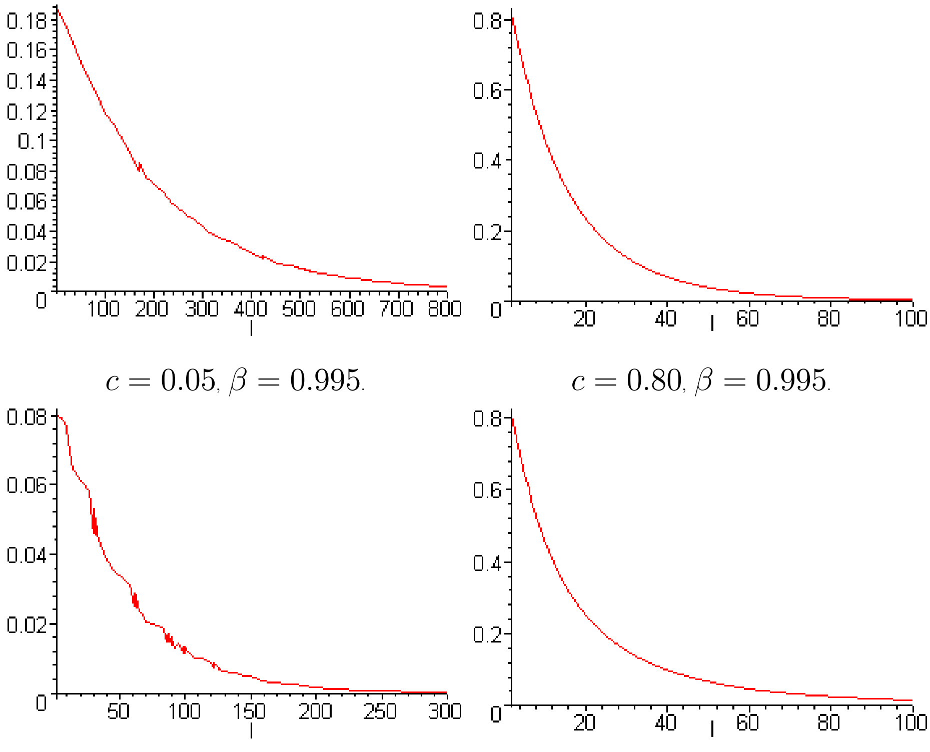
$$B_{\mu_l} = \Gamma(1 + \mu_l)\Gamma(3 - \mu_l)2^{(\mu_l - 1)}, \quad \mu_l = c\beta^{l-1}.$$

Proof. The proof follows from theorem 1, by setting $p = a = \mu = 1$, $q = 3$ and $\mu_l = c\beta^{l-1}$. ■

Figures 1 and 2 plot the theoretical ACF of the duration of the above processes with $(c, \beta) \in \{(0.05, 0.995), (0.80, 0.995), (0.05, 0.98), (0.80, 0.98)\}$ (we used Maple to evaluate the autocorrelations).

Figure 1

Autocorrelation function of x_i for F-EACD$_1$(1,1), $a = 4$

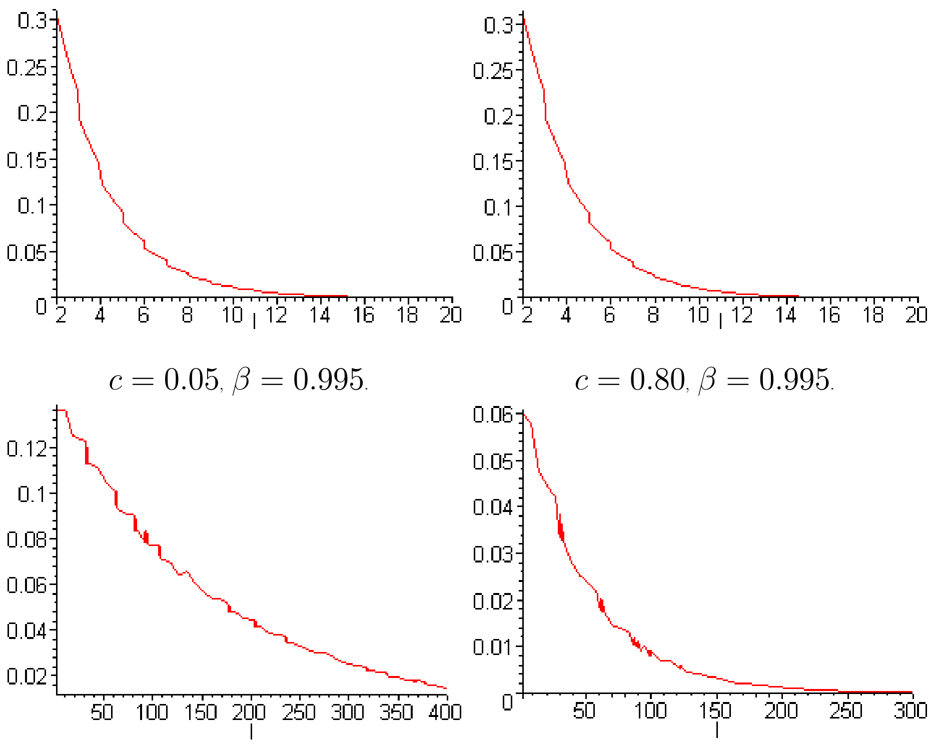


$c = 0.05$, $\beta = 0.995$. $c = 0.80$, $\beta = 0.995$.

$c = 0.05$, $\beta = 0.98$. $c = 0.80$, $\beta = 0.98$.

Figure 2

Autocorrelation function of x_i for L-EACD$_1$(1,1), $q = 3$



$c = 0.05$, $\beta = 0.995$. $c = 0.80$, $\beta = 0.995$.

$c = 0.05$, $\beta = 0.98$. $c = 0.80$, $\beta = 0.98$.

Next, recall that $\Gamma(\cdot)$ indicates the gamma function and p, a are the parameters of the GG distribution (see equation (4)). We have the following theorem.

Theorem 2 *Let assumptions 1 and 2 hold and* $\frac{2\mu}{a}, \frac{2\mu_l}{a} > -p$. *Then the* 2μ*th moment of the duration and the* kth *autocorrelation of the* μ*th power of the duration, for the GG-EACD*$_1(n,m)$ *model in equations (1) and (4)-(6), are given by*

$$\mathsf{E}(x_i^{2\mu}) = e^{\frac{2\mu\omega}{B(1)}}\Gamma_{2\mu}\prod_{l=1}^{\infty}(\Gamma_{2\mu_l}), \tag{13}$$

$$\rho_k(x_i^{\mu}) = \frac{\Gamma_\mu\left\{\Gamma_{\mu+\mu_k}\prod_{r=1}^{k-1}(\Gamma_{\mu_r})\prod_{l=1}^{\infty}(\Gamma_{\mu_{k+l}+\mu_l}) - \Gamma_\mu\left[\prod_{l=1}^{\infty}(\Gamma_{\mu_l})\right]^2\right\}}{\Gamma_{2\mu}\prod_{l=1}^{\infty}(\Gamma_{2\mu_l}) - (\Gamma_\mu)^2\left[\prod_{l=1}^{\infty}(\Gamma_{\mu_l})\right]^2}, \tag{14}$$

with

$$\Gamma_{\mu_l} \triangleq \Gamma(\mu_l) = \frac{\Gamma\left(p+\frac{\mu_l}{a}\right)[\Gamma(p)]^{(\mu_l-1)}}{\left[\Gamma\left(p+\frac{1}{a}\right)\right]^{\mu_l}},$$

where μ_l is *defined in lemma 1.*

To illustrate the general result we consider the EACD$_1$(1,1) process with innovations that are drawn from either the Rayleigh or the exponential distribution. These models will be called Rayleigh EACD$_1$ (R-EACD$_1$) and exponential EACD$_1$ (E-EACD$_1$) respectively. In these cases $\mu_l = \mu c\beta^{l-1}$. We have the following corollaries.

Corollary 2a *For the R-EACD*$_1$*(1,1) model satisfying* $|c|, |\beta| < 1$, *the* k*th autocorrelation of the duration is given by equation (14) with*

$$\Gamma_{\mu_l} = \frac{\Gamma\left(1+\frac{\mu_l}{2}\right)}{\left[\Gamma\left(\frac{3}{2}\right)\right]^{\mu_l}}, \quad \mu_l = c\beta^{l-1}.$$

Proof. The proof follows from theorem 2, by setting $p = \mu = 1$, $a = 2$ and $\mu_l = c\beta^{l-1}$. ■

Corollary 2b *For the E-EACD*$_1$*(1,1) model satisfying* $-0.50 < c < 1$ $(c \neq 0)$, $|\beta| < 1$ *and* $-0.50 < c\beta$ *the* k*th autocorrelation of the duration is given by equation (14) with*

$$\Gamma_{\mu_l} = \Gamma(1+\mu_l), \quad \mu_l = c\beta^{l-1}.$$

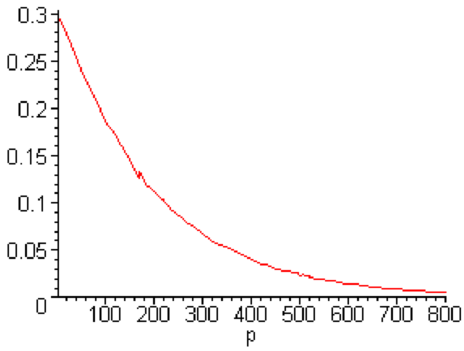
Proof. The proof follows from theorem 2, by setting $p = a = \mu = 1$, and $\mu_l = c\beta^{l-1}$. ■

For the E-EACD$_1$(1,1) model Bauwens and Giot (2001a) illustrate the variation of $\rho_1(x_i)$ as a function of c (from 0 to 0.20) and β (from 0.80 to 0.98).

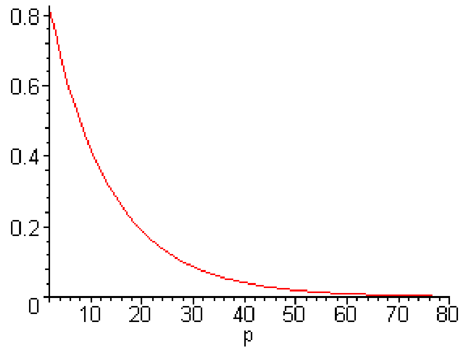
Figures 3 and 4 plot the theoretical ACF of the duration of the above processes with $(c, \beta) \in \{(0.05, 0.995), (0.80, 0.995), (0.05, 0.98), (0.80, 0.98)\}$.

Figure 3.

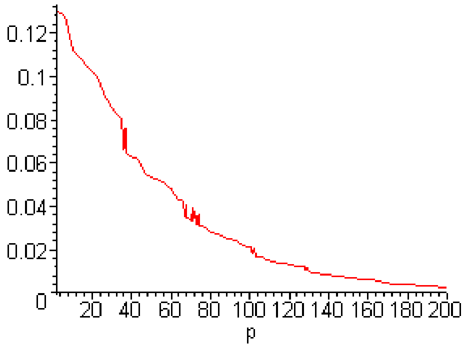
Autocorrelation function of x_i for R-EACD$_2$(1,1)



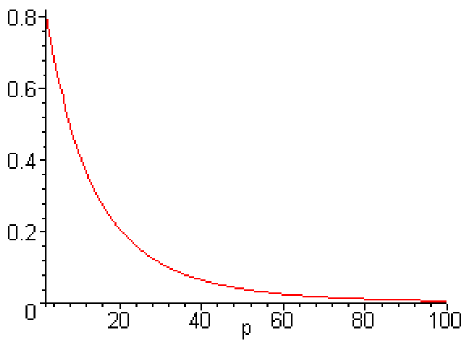
$c = 0.05, \beta = 0.995.$



$c = 0.80, \beta = 0.995.$



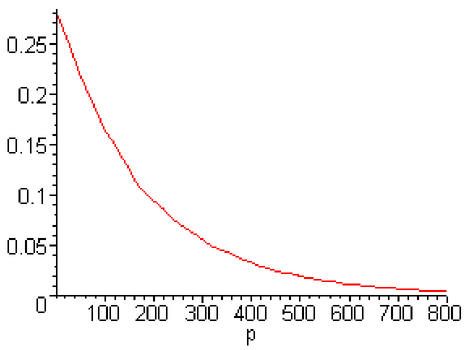
$c = 0.05, \beta = 0.98.$



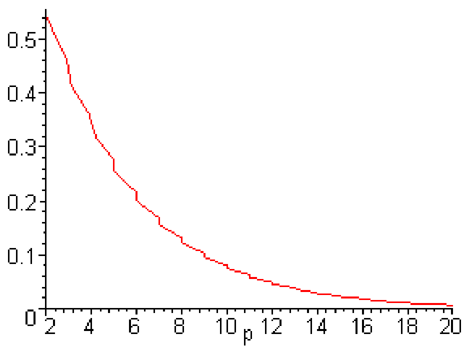
$c = 0.80, \beta = 0.98.$

Figure 4

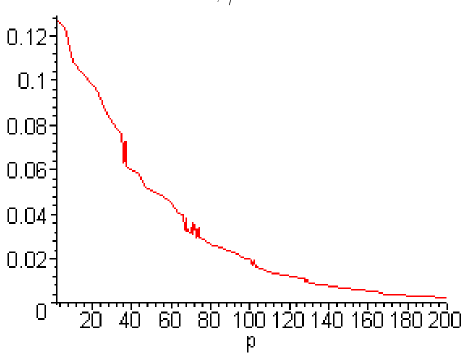
Autocorrelation function of x_i for E-EACD$_2$(1,1)



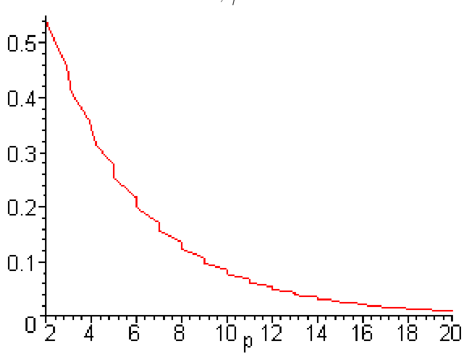
$c = 0.05, \beta = 0.995.$



$c = 0.80, \beta = 0.995.$



$c = 0.05, \beta = 0.98.$



$c = 0.80, \beta = 0.98.$

Theorem 3 *Suppose that assumptions 1 and 2 hold and* $\frac{2\mu}{a} > -p,\ 2\mu_l < \varphi^{-a}$. *Then the* 2μ*th moment of the duration and the* k*th autocorrelation of the* μ*th power of the duration, for the GG-EACD*$_2(m,n)$ *model in equations (1), (4)-(5) and (7), are given by*

$$\mathsf{E}(x_i^{2\mu}) = e^{\frac{2\mu\omega}{B(1)}}\Gamma_{2\mu}\prod_{l=1}^{\infty}(\Delta_{2\mu_l}), \tag{15}$$

$$\rho_k(x_i^{\mu}) = \frac{\Gamma_\mu\left\{\Delta_{\mu,\mu_k}\prod_{r=1}^{k-1}(\Delta_{\mu_r})\prod_{l=1}^{\infty}(\Delta_{\mu_{k+l}+\mu_l}) - \Gamma_\mu\left[\prod_{l=1}^{\infty}(\Delta_{\mu_l})\right]^2\right\}}{\Gamma_{2\mu}\prod_{l=1}^{\infty}(\Delta_{2\mu_l}) - (\Gamma_\mu)^2\left[\prod_{l=1}^{\infty}(\Delta_{\mu_l})\right]^2}, \tag{16}$$

with

$$\Delta_{\mu,\mu_k} \triangleq \Delta(\mu,\mu_k) = \frac{\Gamma\left(p+\frac{\mu}{a}\right)[\Gamma(p)]^{(\mu-1)}[\Gamma\left(p+\frac{1}{a}\right)]^{pa}}{\left\{\left[\Gamma(p+\frac{1}{a})\right]^a - \mu_k[\Gamma(p)]^a\right\}^{\left(p+\frac{\mu}{a}\right)}},$$

$$\Delta_{\mu_l} \triangleq \Delta(\mu_l) = \frac{[\Gamma\left(p+\frac{1}{a}\right)]^{pa}}{\left\{\left[\Gamma(p+\frac{1}{a})\right]^a - \mu_l[\Gamma(p)]^a\right\}^p},$$

where μ_l, *and* Γ_μ *are defined in lemma 1 and theorem 2 respectively, and* p, a *are the parameters of the GG distribution.*

As an illustration we consider the GG-EACD$_2$(1,1) model. Let Φ be the basic hypergeometric series denoted by ${}_1\Phi_0(b;d,z) = \sum_{j=0}^{\infty}\frac{(b;d)_j}{(d;d)_j}z^j$, where $(b;d)_j$ is the d-shifted factorial.

Corollary 3a *For the GG-EACD*$_2$*(1,1) model satisfying* $|\beta| < 1$,$-1 < c < \frac{\varphi^{(-a)}}{2\mu}$ $(c \neq 0)$ *and* $c\beta < \frac{\varphi^{(-a)}}{2\mu}$ *the* k*th autocorrelation of the* μth power of the *duration is given by*

$$\rho_k(x_i^{\mu}) = \frac{\Gamma_\mu\left\{\Delta_{\mu,\mu_k}\left[{}_1\Phi_0(\beta^{k-1};\beta,c^*)\right]^p\left[{}_1\Phi_0(0;\beta,c_k)\right]^p - \Gamma_\mu\left[{}_1\Phi_0(0;\beta,c^*)\right]^p\right\}}{\Gamma_{2\mu}\left[{}_1\Phi_0(0;\beta,2c^*)\right]^p - (\Gamma_\mu)^2\left[{}_1\Phi_0(0;\beta,c^*)\right]^{2p}}, \tag{17}$$

with

$$c^* = \frac{c\mu\left[\Gamma(p)\right]^a}{\left[\Gamma(p+\frac{1}{a})\right]^a}, \quad c_k \triangleq c(k) = c^*(1+\beta^k), \quad \mu_l = \mu c\beta^{l-1},$$

where Γ_μ *and* Δ_{μ,μ_k} *are defined in lemma 1 and theorem 3 respectively.*

Proof. The proof follows from theorem 3, by setting $\mu_l = \mu c\beta^{l-1}$. ■

As a further illustration we consider the R-EACD$_2$ and E-EACD$_2$ processes of order (1,1). We have the following corollaries.

Corollary 3b *For the R-EACD*$_2$*(1,1) model satisfying* $|\beta| < 1$, $-1 < c < 0.393$ $(c \neq 0)$, $c\beta < 0.393$ *the* k*th autocorrelation of the duration is given by equation (17),* *with* $p = \mu = \Gamma_\mu = 1$ *and*

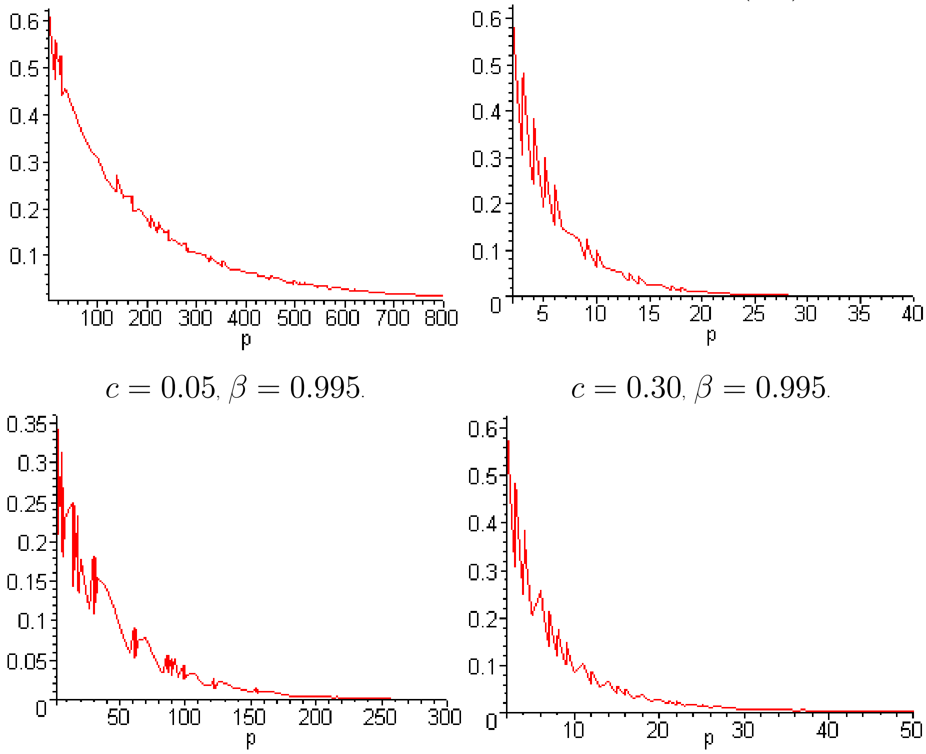
$$\Delta_{1,\mu_k} = \frac{1}{\{1 - 1.273\mu_k\}^{\frac{3}{2}}}, \qquad c^* = 1.273c.$$

Proof. The proof follows from corollary 3a, by setting $p = \mu = 1$, and $a = 2$. ■

Figure 5 plots the theoretical ACF of the duration of the above process with $(c, \beta) \in \{(0.05, 0.995), (0.30, 0.995), (0.05, 0.98), (0.30, 0.98)\}$.

Figure 5

Autocorrelation function of x_i for R-EACD$_2$(1,1)



$c = 0.05$, $\beta = 0.995$. $c = 0.30$, $\beta = 0.995$.

$c = 0.05$, $\beta = 0.98$. $c = 0.30$, $\beta = 0.98$.

Corollary 3c *For the E-EACD$_2$(1,1) model satisfying* $|\beta| < 1$, $-1 < c < 0.50$ $(c \neq 0)$ *and* $c\beta < 0.50$ *the kth autocorrelation of the duration is given by equation (17), with* $p = \mu = \Gamma_\mu = 1$ *and*

$$\Delta_{1,\mu_k} = \frac{1}{(1 - \mu_k)^2}, \qquad c^* = c.$$

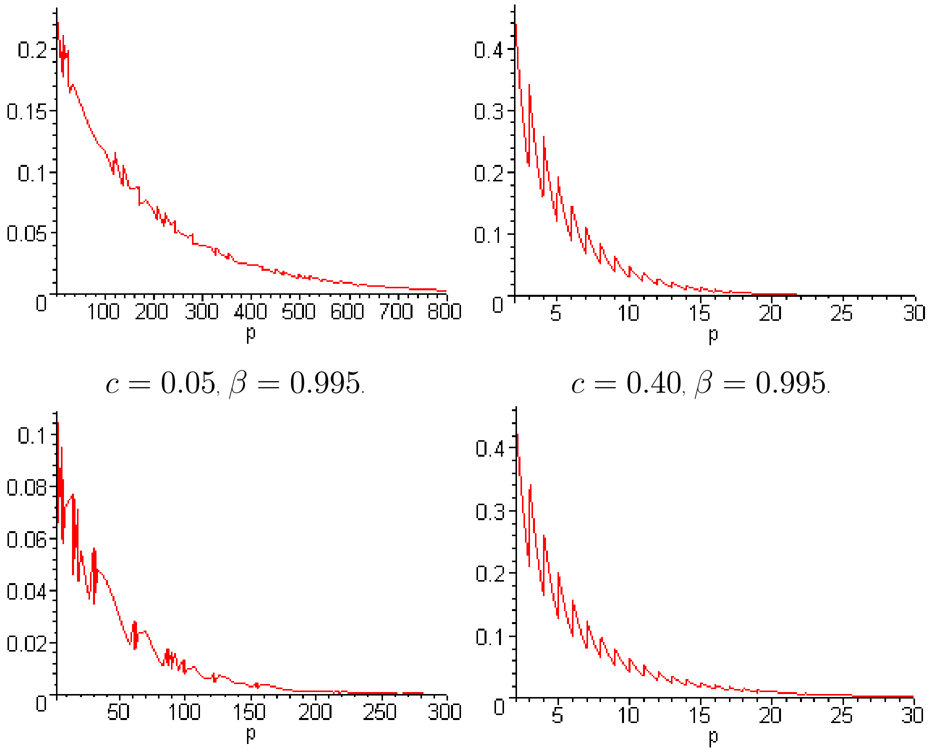
Proof. The proof follows from corollary 3a, by setting $p = a = \mu = 1$. ■

For the E-EACD$_2$(1,1) model Bauwens and Giot (2001a) illustrate the variation of $\rho_1(x_i)$ as a function of c (from 0 to 0.20) and β (from 0.80 to 0.98).

Figure 6 plots the theoretical ACF of the duration of the above process with $(c, \beta) \in \{(0.05, 0.995), (0.40, 0.995), (0.05, 0.98), (0.40, 0.98)\}$.

Figure 6

Autocorrelation function of x_i for E-EACD$_2$(1,1)



$c = 0.05, \beta = 0.995.$ $c = 0.40, \beta = 0.995.$

$c = 0.05, \beta = 0.98.$ $c = 0.40, \beta = 0.98.$

For all the models in corollaries 1-3 the parameters c and β control the decay of the theoretical autocorrelation function. We have the following remarks.

Remark 2a. In all cases it is seen that the autorrelations start higher and decrease more rapidly when the value of c is high than when it is low.

Remark 2b. In all cases, when the value of c is low, the model based autocorrelations seem to start higher and decrease more slowly when the value of β is high than when it is low.

Remark 2c. In all cases, when the value of c is high, it is seen that the autocorrelations start higher and decrease faster when the value of β is high than when it is low.

Until the present contribution and the papers by Bauwens and Giot (2001a), and Bauwens, Galli and Giot (2008), the unconditional moments implied by the logarithmic(exponential) ACD models were not available analytically. Bauwens and Giot (2000) relied therefore on numerical simulations to compute these moments.

4 Conclusions

This paper has provided a detailed description of the various EACD models. We also investigated the properties of the GG and GF distributions, which allow for non-monotonic hazard functions. For all ACD specifications we derived analytical expressions of the ACF of the durations. Conditions for the existence of the first two moments of the durations were also established. The derivation of the autocorrelations of the durations and their comparison with the corresponding sample equivalents will help the investigator (a) to decide which is the most appropriate method of estimation (e.g. maximum likelihood estimation, minimum distant estimator) for a specific model, (b) to choose, for a given estimation technique, the model (e.g. LMACD, EACD_1, EACD_2) that best replicates certain stylized facts about the data and, (c) in conjunction with the various model selection criteria, to identify the optimal order of the chosen specification.

Appendix

Proof. [Lemma 1] The logarithm of ψ_i in equation (5) can be expressed as an infinite distributed lag of $f(\varepsilon_i)$ terms:

$$\ln(\psi_i) = \frac{\omega}{B(1)} + \frac{C(L)}{B(L)} f(\varepsilon_i).$$

The above expression can be written as

$$\ln(\psi_i) = \frac{\omega}{B(1)} + \sum_{l=1}^{\infty} \delta_l f(\varepsilon_{i-l}), \tag{A.1}$$

where

$$\delta_l = \sum_{j=1}^{n} \zeta_j z_{jl},$$

with

$$\begin{aligned} \zeta_j &= \frac{\lambda_j^{n-1}}{\prod_{f=1, f\neq j}^{n} (\lambda_j - \lambda_f)}, \\ z_{jl} &\equiv \begin{cases} \sum_{r=0}^{l-1} c_{l-r} \lambda_j^r, & \text{if} \quad l \leq m, \\ z_{jm} \lambda_j^{l-m}, & \text{if} \quad l > m \end{cases}. \end{aligned}$$

From equation (A.1) it follows that

$$\psi_i = e^{\frac{\omega}{B(1)}} \prod_{l=1}^{\infty} [e^{\delta_l f(\varepsilon_{i-l})}].$$

Raising both sides of the above equation to power μ and using the fact that $x_i^\mu = \psi_i^\mu \varepsilon_i^\mu$ yields equation (8). ■

Proof. [Lemma 2] Rewriting (8) we have

$$x_i^\mu = e^{\frac{\mu\omega}{B(1)}} \varepsilon_i^\mu \prod_{l=1}^{\infty} [e^{\mu_l f(\varepsilon_{i-l})}], \tag{A.2a}$$

or

$$x_{i-k}^\mu = e^{\frac{\mu\omega}{B(1)}} \varepsilon_{i-k}^\mu \prod_{l=1}^{\infty} [e^{\mu_l f(\varepsilon_{i-k-l})}], \tag{A.2b}$$

where $\mu_l = \mu\delta_l$.

Multiplying equation (A.2a) by (A.2b) and taking expectations yields

$$\begin{aligned} \mathsf{E}(x_i^\mu x_{i-k}^\mu) &= e^{\frac{2\mu\omega}{B(1)}} \mathsf{E}(\varepsilon_i^\mu) \mathsf{E}(\psi_i^\mu \varepsilon_{i-k}^\mu \psi_{i-k}^\mu) = \\ &= e^{\frac{2\mu\omega}{B(1)}} \mathsf{E}(\varepsilon_i^\mu) \mathsf{E}\left[\varepsilon_{i-k}^\mu e^{\mu_k f(\varepsilon_{i-k})}\right] \prod_{r=1}^{k-1} [\mathsf{E}(e^{\mu_r f(\varepsilon_{i-r})})] \prod_{l=1}^{\infty} [\mathsf{E}(e^{(\mu_{k+l}+\mu_l) f(\varepsilon_{i-k-l})})]. \end{aligned}$$

Using the above expression and the fact that $\rho_k(x_i^\mu) = \frac{\mathsf{E}(x_i^\mu x_{i-k}^\mu) - [\mathsf{E}(x_i^\mu)]^2}{\mathsf{E}(x_i^{2\mu}) - [\mathsf{E}(x_i^\mu)]^2}$ we obtain equation (10). ■

Proof. [Theorem 1] Recall that for the EACD$_1$ model, we have

$$f(\varepsilon_i) = \ln(\varepsilon_i). \tag{A.3}$$

In addition, the μth moment for the GF distribution is

$$\mathsf{E}(\varepsilon_i^\mu) = \frac{B(p+\frac{\mu}{a}, q-\frac{\mu}{a}) B(p,q)^{\mu-1}}{B\left(p+\frac{1}{a}, q-\frac{1}{a}\right)^\mu} \quad \forall\ i, \tag{A.4}$$

where p, q and a are the parameters of the GF distribution (see equation (3)). Inserting equations (A.3)-(A.4) into equations (9)-(10) yields equations (11)-(12). ■

Proof. [Theorem 2] When the innovations $\{\varepsilon_i\}$ are drawn from the GG distribution, we have

$$\mathsf{E}(\varepsilon_i^\mu) = \frac{\Gamma(p+\frac{\mu}{a}) \Gamma(p)^{\mu-1}}{\Gamma\left(p+\frac{1}{a}\right)^\mu} \quad \forall\ i, \tag{A.5}$$

where p and a are the parameters of the GG distribution (see equation (4)). Inserting equations (A.3) and (A.5) into equations (9)-(10) yields equations (13)-(14). ■

Proof. [Theorem 3] Recall that for the GG-EACD$_2$ model, we have

$$f(\varepsilon_i) = \varepsilon_i^a, \tag{A.6}$$

where a is one of the parameters of the GG distribution. Further, by direct computation we obtain

$$\mathsf{E}(\varepsilon_i^\mu e^{\mu_k \varepsilon_i^a}) = \frac{\Gamma(p+\frac{\mu}{a})[\Gamma(p)]^{(\mu-1)}\left[\Gamma\left(p+\frac{1}{a}\right)\right]^{pa}}{\left\{\left[\Gamma\left(p+\frac{1}{a}\right)\right]^a - \mu_k[\Gamma(p)]^a\right\}^{\left(p+\frac{\mu}{a}\right)}} \quad \forall\ i. \tag{A.7}$$

Note that equation (A.7), when $\mu_k = 0$, gives equation (A.5). Inserting equations (A.5)-(A.7) into equations (9)-(10) yields equations (15)-(16). ■

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