Quantitative and Qualitative Analysis in Social Sciences
Volume 1, Issue 2, 2007, 71-87
ISSN: 1752-8925

The Correlation Structure of Some Financial Time Series Models

Menelaos Karanasos[a]
Brunel University

Abstract

The purpose of this paper is to examine the correlation structure of mixed autoregressive and moving average (ARMA) models, as discussed in Granger and Morris (1976). The technique we use to obtain the autocorrelations is based on the Wold representation of an ARMA specification as given in Pandit (1973) or in Karanasos (2001). We give two examples to illustrate our general results: (i) two ARMA(2,2) processes with identical autoregressive polynomials and different moving average ones, and (ii) two ARMA(2,1) formulations with different autoregressive and moving average polynomials. The techniques in this paper can be employed to calculate the cross-correlations between the cross-products of the residuals in multivariate nonlinear time series models, e.g. GARCH and Markov Switching models.

JEL Classifications: C22, C32.
Keywords: Autocorrelations, mixed ARMA models, Wold representation, aggregation.

Acknowledgements: We would like to thank M. Dietrich and M. Karanassou for their helpful comments and suggestions.

[a] Economics and Finance, Brunel University, West London, UB8 3PH, UK; tel.: +44 (0)1895 265284, email: menelaos.karanasos@brunel.ac.uk

1 Introduction

The purpose of this paper is to examine the correlation structure of mixed ARMA time series models (introduced by Yule, 1921, 1927).

The theoretical autocovariance (acv) function of the ARMA process is an important tool in time series analysis. The calculation of the theoretical autocovariances is used (i) to estimate ARMA formulations with conventional exact maximum likelihood (ML) procedures (ii) to analyze the distribution of estimated ARMA parameters (Hannan, 1970) and (iii) to initialize simulations with ARMA models. The theoretical acf of the univariate ARMA(p, q) specification has already been derived in the literature. The autocovariances are obtained in terms of the roots of the autoregressive (AR) polynomial and the parameters of the moving average (MA) one.

Pandit (1973, pp. 100, 141) and Pandit and Wu[1] (1983, pp. 105, 129-130), derived the acv function of the ARMA(p, q) model (when the roots of the AR polynomial are distinct) by using its infinite moving average (ima) or Wold representation.[2] Nerlove, Grether and Carvalho (1979, pp. 39, 78-85) used the canonical factorization (cf) of the autocovariance generating function (agf) to derive a general expression for the acf of the ARMA(p, q) process (as an illustration they presented the acv function of the MA(q), AR(p), and ARMA(1,1) specifications).[3]

Zinde-Walsh (1988) obtained the acv function of the ARMA(p, q) model by using its standard (see for example, Priestley, 1981) spectral representation.[4] She derived expressions for both cases of simple and multiple roots of the AR polynomial. Karanasos

[1] Pandit and Wu (1983) examine only the case where $q < p$.

[2] They called the coefficient function in this expansion Green's function (see Miller, 1968). What Pandit and Wu (1983) have called Green's function is also referred to in the literature as weighting function (see Wiener, 1949, and Pugachev, 1957) or as ψ weights (see Box and Jenkins, 1970).

For a proof of Wold's theorem see Wold (1938, pp. 75-89), Hannan (1970, pp. 136-137), Anderson (1971, pp. 420-421), Sargent (1979, pp. 257-262) or Brockwell and Davis (1987, pp. 180-182); also see Hannan (1970, pp. 157-158) for the vector case. Wold's theorem has also been discussed by, for example, Nerlove, Grether and Carvalho (1979, pp. 30-36), Pandit and Wu (1983, pp. 87-89), and Reinsel (1993, p. 7).

The ima representation of the ARMA model has been discussed by, for example, Anderson (1976, p. 44), Pandit and Wu (1983, p. 108), Granger and Newbold (1986, pp. 25-26), Brockwell and Davis (1987, pp. 87-89), Wei (1989, p.56), Hamilton (1994, pp. 59-60), Gourieroux and Monfort (1997, pp. 160-162); see also Mittnik (1987), Lutkepohl (1993, pp. 220-221), Reinsel (1993, pp. 33-34) or Gourieroux and Monfort (1997, p. 252) for the vector case.

[3] The cf of the agf has also been discussed by, for example, Anderson (1976, p. 129), Granger and Newbold (1986, pp. 26-27), Brockwell and Davis (1987, pp. 102-103), Wei (1989, pp. 242-243), or Hamilton (1994, pp. 61-63); see also Brockwell and Davis (1987, p. 410) or Reinsel (1993, pp. 33-34), for the vector case.

[4] For the important subject of spectral analysis see also the excellent books by Hannan (1967) and Jenkins and Watts (1968).

(1998) derived the acv function of the specification of order (p,q) (when the roots of the AR polynomial are distinct) by expressing it as an AR(1) process with an ARMA$(p-1,q)$ error. Algorithms for computing the theoretical autocovariances for univariate ARMA formulations have been suggested in McLeod (1975) and Tunnicliffe Wilson (1979).

This paper contributes to the above literature by deriving the acv function of the sum of ARMA specifications when the roots of the AR polynomials are distinct. Granger and Morris (1976) consider such mixed ARMA models. As most economic series are both aggregates and are measured with error it follows that these mixed processes are often found in practice. To facilitate model identification, the results for the autocorrelations of the aggregate process can be applied so that properties of the observed data can be compared with the theoretical properties of the models.

We examine three distinct cases, depending on whether the processes have identical or different autoregressive parts and uncorrelated or correlated innovations. We obtain our results by using the ima representation of the ARMA(p,q) model, as given in Pandit (1973) or in Karanasos (2001).[5] The autocovariances are expressed in terms of the roots of the AR polynomials and the parameters of the MA ones. In the case of distinct roots the results in Zinde-Walsh (1988) or Karanasos (1998) are special cases of our Theorem 1 (see Proposition 1).

2 Mixed ARMA Models

Let y_t be an ARMA(p,q) process given by

$$\Phi(L)y_t = \mu + \Theta(L)\epsilon_t, \tag{1}$$

with

$$\Theta(L) = \sum_{k=0}^{q} \theta_k L^k,$$
$$\Phi(L) = -\sum_{l=0}^{p} \phi'_l L^l = \prod_{l=1}^{p}(1-\phi_l L), \quad \phi'_0 = -1,$$

where $\{\epsilon_t\}$ is a sequence of identical independent distributed (i.i.d) random variables with mean zero and finite variance σ^2. Notice that, $\phi_l\,(l=1,\ldots,p)$ denotes the inverse

[5] Karanasos (2001) has also used the ima representation in the context of the GARCH and GARCH in mean models.

of the lth root of $\Phi(L)$.

Assumption 1 (Stationarity) *All the roots of the AR polynomial $\Phi(L)$ lie outside the unit circle.*

Assumption 2 (Identifiability) *The polynomials $\Phi(L)$ and $\Theta(L)$ are left coprime. In other words the representation $\frac{\Phi(L)}{\Theta(L)}$ is irreducible.*

In what follows we examine only the case where the roots of the AR polynomial $\Phi(L)$ are distinct ($\phi_r \neq \phi_l$ for $r, l = 1, \ldots, p$, and $r \neq l$). Our first proposition establishes the lag-m autocorrelation (acr) of y_t, $\rho_m(y_t) = \mathrm{Corr}(y_t, y_{t-m})$, $m \,\epsilon\, \mathbb{N}$.

Proposition 1. Under Assumptions 1 and 2, the acr function of y_t is given by

$$\rho_m(y_t) = \frac{\gamma_m}{\gamma_0}, \ \ \gamma_m = \sum_{l=1}^{p} \zeta_{lm} \lambda_{lm^\star} \sigma^2, \tag{2}$$

with

$$\zeta_{lm} = \frac{\phi_l^{p-1+m}}{\prod\limits_{r=1}^{p} (1 - \phi_l \phi_r) \prod\limits_{\substack{r=1 \\ r \neq l}}^{p} (\phi_l - \phi_r)},$$

and

$$\lambda_{lm} = \sum_{k=0}^{q} \theta_k^2 + \sum_{d=1}^{m^\star} \sum_{\kappa=0}^{q-d} \theta_\kappa \theta_{\kappa+d} (\phi_l^d + \phi_l^{-d}) + \sum_{d=m^\star+1}^{q} \sum_{\kappa=0}^{q-d} \theta_\kappa \theta_{\kappa+d} (\phi_l^d + \phi_l^{d-2m}),$$

where $m^\star = \min(m, q)$.[6]

Proof. See Appendix A.

By considering the model generating the sum of two or more series, Granger and Morris (1976) have shown that the mixed ARMA model is the one most likely to occur. They mentioned that "two situations where series are added together are of particular interpretational importance. The first is where series are aggregated to form some total, and the second one is where the observed series is the sum of the true process plus an observational error. Most of the macroeconomic series, such as GNP, unemployment or exports are aggregates. Virtually any macroeconomic series, other than certain prices or interest rates, contain important observation errors." Accordingly the following analysis

[6] Zinde Victoria Walsh (1988) derived the above result (eq. 2) by using the standard (see for example, Priestley, 1981) spectral representation for the acv function of a stationary ARMA process whereas Karanasos (1998) derived the same result by expressing the ARMA(p,q) model as an AR(1) process with an ARMA(p-1,q) error. Zinde-Walsh's formula is general as it is not restricted to the case of distinct roots.

of the acv function of the sum of ARMA processes can be a useful tool for the applied economist.

Next, we consider the situation where the stochastic process y_t is a simple linear aggregate of n processes y_{it}, i.e.,

$$y_t = \sum_{i=1}^{n} y_{it}. \tag{3}$$

Each individual process y_{it} is assumed to be an ARMA(p, q_i) model satisfying

$$\Phi(L)y_{it} = \mu_i + \Theta_i(L)\epsilon_{it}, \tag{4}$$

with

$$\Theta_i(L) = \sum_{k=0}^{q_i} \theta_{ik} L^k,$$

where $\Phi(L)$ is defined by equation (1). Further, we assume that the innovation vector $\boldsymbol{\epsilon}_t = [\epsilon_{it}]_{i=1,\dots,n}$ is i.i.d with mean vector zero and diagonal (positive definite) covariance matrix $\boldsymbol{\Sigma} = diag\{\boldsymbol{\sigma}\}$ where $\boldsymbol{\sigma} = [\sigma_{ii}]_{i=1,\dots,n}$.

Assumption 3 (Identifiability) *The polynomials $\Phi(L)$ and $\Theta_i(L)$, $i = 1, \dots, n$, are left coprime. In other words the representation $\frac{\Phi(L)}{\Theta_i(L)}$ is irreducible.*

Proposition 2. Let Assumptions 1 and 3 hold. The acr function of y_t is given by

$$\rho_m(y_t) = \frac{\gamma_m}{\gamma_0}, \;\; \gamma_m = \sum_{i=1}^{n}\sum_{l=1}^{p} \zeta_{lm}\lambda_{il,m_i^\star}\sigma_{ii}, \quad m \,\epsilon\, \mathbb{N}, \tag{5}$$

with

$$\lambda_{il,m} = \sum_{k=0}^{q_i}\theta_{ik}^2 + \sum_{d=1}^{m_i^\star}\sum_{\kappa=0}^{q_i-d}\theta_{i\kappa}\theta_{i\kappa+d}(\phi_l^d + \phi_l^{-d}) + \sum_{d=m_i^\star+1}^{q_i}\sum_{\kappa=0}^{q_i-d}\theta_{i\kappa}\theta_{i\kappa+d}(\phi_l^d + \phi_l^{d-2m}),$$

where $m_i^\star = \min(m, q_i)$ and ζ_{lm} is defined in equation (2).

Proof. See Appendix A.

In the proposition that follows the assumption of independence is weakened to allow for contemporaneous correlation between the noise series.

Let y_t be a simple linear aggregate of n processes y_{it} defined by equations (3) and (4). In addition, we assume that the (positive definite) covariance matrix $\boldsymbol{\Sigma}$ is given by $\boldsymbol{\Sigma} = [\sigma_{ij}]_{i,j=1,\dots,n}$.

Proposition 3. If Assumptions 1 and 3 hold, then the acr function of y_t is given by

$$\rho_m(y_t) = \frac{\gamma_m}{\gamma_0}, \quad \gamma_m = \sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{l=1}^{p} \zeta_{lm}\lambda_{ij,lm}\sigma_{ij}, \quad m \,\epsilon\, \mathbb{N}, \tag{6}$$

with

$$\lambda_{ij,lm} = \sum_{c=0}^{q_j}\sum_{d=0}^{q_i'} \theta_{id}\theta_{j,d+c}\phi_l^c + \sum_{c=1}^{m_i^\star}\sum_{d=0}^{q_j'} \theta_{jd}\theta_{i,d+c}\phi_l^{-c} + \sum_{c=m_i^\star+1}^{q_i}\sum_{d=0}^{q_j'} \theta_{jd}\theta_{i,d+c}\phi_l^{c-2m},$$

where $q_i' = min(q_i, q_j - c)$, $q_j' = min(q_j, q_i - c)$, and ζ_{lm}, $m_i^\star$ are defined in equations (2) and (5) respectively.

Proof. See Appendix A.

A potential motivation for the derivation of the results in Proposition 3 is that the autocorrelations of the stochastic process y_t in equation (3) can be used to estimate the parameters in equation (4). The approach is to use the minimum distance estimator (MDE), which estimates the parameters by minimizing the Mahalanobis generalized distance of a vector of sample autocorrelations from the corresponding population autocorrelations (see Baillie and Chung, 2001).

As an illustration, consider two ARMA(2,2) processes y_{1t} and y_{2t}

$$(1-\phi_1 L)(1-\phi_2 L)y_{1t} = \mu_1 + (1+\theta_{11}L + \theta_{12}L^2)\epsilon_{1t}, \tag{7a}$$

$$(1-\phi_1 L)(1-\phi_2 L)y_{2t} = \mu_2 + (1+\theta_{21}L + \theta_{22}L^2)\epsilon_{2t}, \tag{7b}$$

with

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \sim \text{i.i.d}\,(0, \mathbf{\Sigma}), \quad \mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}.$$

Further, let assumptions 1 and 3 hold and assume that $\phi_1 \neq \phi_2$ and $\mathbf{\Sigma}$ is positive definite.

The cross-covariance between y_{1t} and y_{2t} is given by

$$\text{Cov}(y_{1t}, y_{2t}) = \sigma_{12}\{ \frac{\phi_1[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22}+\theta_{11}+\theta_{21}\theta_{12})\phi_1+(\theta_{12}+\theta_{22})\phi_1^2]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$

$$+\frac{\phi_2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22}+\theta_{11}+\theta_{21}\theta_{12})\phi_2+(\theta_{12}+\theta_{22})\phi_2^2]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\}.$$

Moreover, the cross-covariances between the two processes, at lag one, are given by

$$\mathrm{Cov}(y_{1t}, y_{2,t-1}) = \sigma_{12}\{ \frac{\phi_1^2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22})\phi_1+(\theta_{11}+\theta_{21}\theta_{12})\phi_1^{-1}+\theta_{22}\phi_1^2+\theta_{12}\phi_1^0]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$

$$+\frac{\phi_2^2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22})\phi_2+(\theta_{11}+\theta_{21}\theta_{12})\phi_2^{-1}+\theta_{22}\phi_2^2+\theta_{12}\phi_2^0]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\},$$

$$\mathrm{Cov}(y_{2t}, y_{1,t-1}) = \sigma_{12}\{ \frac{\phi_1^2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{11}+\theta_{21}\theta_{12})\phi_1+(\theta_{21}+\theta_{11}\theta_{22})\phi_1^{-1}+\theta_{12}\phi_1^2+\theta_{22}\phi_1^0]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$

$$+\frac{\phi_2^2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{11}+\theta_{21}\theta_{12})\phi_2+(\theta_{21}+\theta_{11}\theta_{22})\phi_2^{-1}+\theta_{12}\phi_2^2+\theta_{22}\phi_2^0]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\}.$$

Finally, the cross-covariances between y_1 and y_2, at lag m ($m \geq 2$), are

$$\mathrm{Cov}(y_{1t}, y_{2,t-m}) = \sigma_{12}\{ \frac{\phi_1^{1+m}[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22})\phi_1+(\theta_{11}+\theta_{21}\theta_{12})\phi_1^{-1}+\theta_{22}\phi_1^2+\theta_{12}\phi_1^{-2}]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$

$$+\frac{\phi_2^{1+m}[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22})\phi_2+(\theta_{11}+\theta_{21}\theta_{12})\phi_2^{-1}+\theta_{22}\phi_2^2+\theta_{12}\phi_2^{-2}]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\},$$

$$\mathrm{Cov}(y_{2t}, y_{1,t-m}) = \sigma_{12}\{ \frac{\phi_1^{1+m}[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{11}+\theta_{21}\theta_{12})\phi_1+(\theta_{21}+\theta_{11}\theta_{22})\phi_1^{-1}+\theta_{12}\phi_1^2+\theta_{22}\phi_1^{-2}]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$

$$+\frac{\phi_2^{1+m}[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{11}+\theta_{21}\theta_{12})\phi_2+(\theta_{21}+\theta_{11}\theta_{22})\phi_2^{-1}+\theta_{12}\phi_2^2+\theta_{22}\phi_2^{-2}]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\}.$$

Note that, for $\theta_{11} = \theta_{21}$, $\theta_{12} = \theta_{22}$ and $\epsilon_{1t} = \epsilon_{2t}$, the two processes are identical ($y_{1t} = y_{2t} = y_t$) and the above expressions give the autocovariances of the y_t process. That is

$$\mathrm{Cov}(y_{1t}, y_{2,t-m}) = \mathrm{Cov}(y_{2t}, y_{1,t-m}) = \mathrm{Cov}_m(y_t).$$

In the following theorem the assumption of identical autoregressive parts is relaxed.

We consider the stochastic process y_t defined by equation (3) and we assume that each individual process y_{it} follows an ARMA(p_i, q_i) process

$$\Phi_i(L)y_{it} = \mu_i + \Theta_i(L)\epsilon_{it}, \tag{8}$$

with

$$\Phi_i(L) = -\sum_{l=0}^{p_i} \phi'_{il} L^l = \prod_{l=1}^{p_i}(1 - \phi_{il}L),$$

where $\Theta_i(L)$ and the innovation vector $\boldsymbol{\epsilon}_t = [\epsilon_{it}]_{i=1,\dots,n}$ are defined in equation (4). Finally, ϕ_{il} ($l = 1, \dots, p_i$) denotes the inverse of the lth root of $\Phi_i(L)$.

Assumption 4 (Stationarity condition) *All the roots of the autoregressive polynomials $\Phi_i(L)$, $i = 1, \dots, n$, lie outside the unit circle.*

Assumption 5 (Identifiability condition) *The polynomials* $\Phi_i(L)$ *and* $\Theta_i(L)$, $i = 1, \ldots, n$, *are left coprime. In other words the representation* $\frac{\Phi_i(L)}{\Theta_i(L)}$ *is irreducible.*

In what follows we examine only the case where the roots of the autoregressive polynomial $\Phi_i(L)$ are distinct ($\phi_{ir} \neq \phi_{il}$, for $r \neq l$, $i = 1, \ldots, n$).

Theorem 1. Under Assumptions 4 and 5, the acr function of y_t *is*

$$\rho_m(y_t) = \frac{\gamma_m}{\gamma_0}, \;\; \gamma_m = \sum_{j=1}^{n}\sum_{i=1}^{n}(\sum_{l=1}^{p_i}\zeta_{il,jm}\lambda_{ij,lm} + \sum_{r=1}^{p_j}\zeta_{jr,im}\lambda_{rm,ij})\sigma_{ij}, \quad m \,\epsilon\, \mathbb{N}, \tag{9}$$

with

$$\zeta_{il,jm} = \frac{\zeta_{il,m}}{\prod_{r=1}^{p_j}(1-\phi_{il}\phi_{jr})},$$

$$\zeta_{il,m} = \frac{\phi_{il}^{p_i-1+m}}{\prod_{\substack{r=1\\ r\neq l}}^{p_i}(\phi_{il}-\phi_{ir})},$$

and

$$\lambda_{ij,lm} = \sum_{c=0}^{q_j}\sum_{d=0}^{q_i'}\theta_{id}\theta_{j,d+c}\phi_{il}^{c} + \sum_{c=1}^{m_i^\star}\sum_{d=0}^{q_j'}\theta_{jd}\theta_{id+c}\phi_{il}^{-c},$$

$$\lambda_{rm,ij} = \sum_{c=m_i^\star+1}^{q_i}\sum_{d=0}^{q_j'}\theta_{jd}\theta_{id+c}\phi_{jr}^{c-2m},$$

where $m_i^\star$, q_i' *and* q_j' *are defined in equations (5) and (6) respectively.*

Proof. See Appendix B.

The derivation of the autocorrelations of mixed ARMA processes and their comparison with the corresponding sample equivalents will help the investigator (a) to decide which is the most appropriate method of estimation (e.g., ML estimation, MDE) for a specific model, (b) to choose, for a given estimation technique, the model that best replicates certain stylized facts of the data and, (c) in conjunction with the various model selection criteria, to identify the optimal order of the chosen specification.

As an illustration, consider two ARMA(2,1) processes

$$(1-\phi_{11}L)(1-\phi_{12}L)y_{1t} = (1+\theta_{11}L)\epsilon_{1t}, \tag{10a}$$
$$(1-\phi_{21}L)(1-\phi_{22}L)y_{2t} = (1+\theta_{21}L)\epsilon_{2t}, \tag{10b}$$

where

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \sim \text{i.i.d}\,(0,\mathbf{\Sigma}), \quad \mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}.$$

Let assumptions 4 and 5 hold and assume that $\phi_{i1} \neq \phi_{i2}$, $i = 1, 2$, and $\mathbf{\Sigma}$ is positive definite.

The cross-covariance between y_{1t} and y_{2t} is

$$\mathrm{Cov}(y_{1t}, y_{2t}) = \sigma_{12}\{ \tfrac{\phi_{11}[1+\theta_{11}\theta_{21}+\theta_{21}\phi_{11}]}{(\phi_{11}-\phi_{12})(1-\phi_{11}\phi_{21})(1-\phi_{11}\phi_{22})} + \tfrac{\phi_{12}[1+\theta_{11}\theta_{21}+\theta_{21}\phi_{12}]}{(\phi_{12}-\phi_{11})(1-\phi_{12}\phi_{21})(1-\phi_{12}\phi_{22})} +$$

$$+ \tfrac{\phi_{21}^2\theta_{11}}{(\phi_{21}-\phi_{22})(1-\phi_{21}\phi_{11})(1-\phi_{21}\phi_{12})} + \tfrac{\phi_{22}^2\theta_{11}}{(\phi_{22}-\phi_{21})(1-\phi_{22}\phi_{11})(1-\phi_{22}\phi_{12})}\}.$$

Moreover, the covariances between the two processes, at lag m ($m \geq 1$), are given by

$$\mathrm{Cov}(y_{1t}, y_{2,t-m}) = \sigma_{12}\{ \tfrac{\phi_{11}^{1+m}[1+\theta_{11}\theta_{21}+\theta_{21}\phi_{11}+\theta_{11}\phi_{11}^{-1}]}{(\phi_{11}-\phi_{12})(1-\phi_{11}\phi_{21})(1-\phi_{11}\phi_{22})} + \tfrac{\phi_{12}^{1+m}[1+\theta_{11}\theta_{21}+\theta_{21}\phi_{12}+\theta_{11}\phi_{12}^{-1}]}{(\phi_{12}-\phi_{11})(1-\phi_{12}\phi_{21})(1-\phi_{12}\phi_{22})}\},$$

and

$$\mathrm{Cov}(y_{2t}, y_{1,t-m}) = \sigma_{12}\{ \tfrac{\phi_{21}^{1+m}[1+\theta_{11}\theta_{21}+\theta_{11}\phi_{21}+\theta_{21}\phi_{21}^{-1}]}{(\phi_{21}-\phi_{22})(1-\phi_{21}\phi_{11})(1-\phi_{21}\phi_{12})} + \tfrac{\phi_{22}^{1+m}[1+\theta_{11}\theta_{21}+\theta_{11}\phi_{22}+\theta_{21}\phi_{22}^{-1}]}{(\phi_{22}-\phi_{21})(1-\phi_{22}\phi_{11})(1-\phi_{22}\phi_{12})}\}.$$

Note that, when $\phi_{11} = \phi_{21} = \phi_1$, $\phi_{12} = \phi_{22} = \phi_2$ and $\theta_{12} = \theta_{22} = 0$ in equations (10) and (7), respectively, the corresponding expressions for the cross-covariances are equal.

Finally, we should mention that the techniques in this paper can be employed to calculate the cross-correlations between the cross-products of the residuals in multivariate nonlinear time series models. As economic and financial variables are inter-related, generalization of univariate formulations to the multivariate set-up is quite natural, in particular for the GARCH models. These nonlinear processes have been widely used in empirical applications. More specifically, they have been used by: Bollerslev, Engle and Wooldridge (1988) for their analysis of returns, bills, bonds and stocks; Baillie and Bollerslev (1990) to model risk premia in forward foreign exchange rate markets; Baillie and Myers (1991) to estimate optimal hedge ratios in commodity markets; Kroner and Claessens (1991) to analyze the optimal currency composition of external debt; Ng (1991) to test the CAPM; McCurby and Morgan (1991) to test the uncovered interest rate parity; Karolyi (1995) to analyze stock returns; Grier and Perry (2000) to examine the effects of real and nominal uncertainty on inflation and output growth. The extensive use of these processes in modelling conditional volatility in high frequency financial assets demonstrates the popularity of the various multivariate GARCH models (see, for example, the excellent surveys by Shephard, 1996, and Pagan, 1996).

3 Concluding Remarks

The purpose of this paper was to examine the correlation structure of mixed ARMA models. In Section 2 we presented the acr function of the sum of n ARMA(p_i, q_i) processes using the ima representation of an ARMA specification. The autocorrelations were expressed in terms of the roots of the AR polynomials and the parameters of the MA polynomials. In this paper we examined only the case of distinct roots of the AR polynomials-the case of equal roots is left for future research. The ima representation technique can also be applied to obtain the correlation structure of non linear multivariate time series models, e.g. GARCH and Markov Switching modelios.

Appendix A

PROOF OF PROPOSITION 1

The impulse response function (irf) of the ARMA(p, q) process y_t in equation (1) is given by

$$y_t = \mu + \sum_{\tau=0}^{\infty} e_\tau \epsilon_{t-\tau}, \tag{A.1}$$

where

$$e_\tau = \sum_{l=1}^{p} \sum_{\kappa=0}^{\min(\tau,q)} \tilde{\zeta}_{l,\tau-\kappa}\theta_\kappa,$$

and

$$\tilde{\zeta}_{lm} = \frac{\phi_l^{p-1+m}}{\prod\limits_{\substack{r=1 \\ r\neq l}}^{p} (\phi_l - \phi_r)},$$

(see Pandit, 1973 or Karanasos, 2001).

Multiplying $(y_t - \mu)$ by $(y_{t-m} - \mu)$ $(m \in \mathbb{N})$ and taking expected values gives the lag-m acv of y_t :

$$\mathrm{Cov}_m(y_t) = \sum_{\tau=0}^{\infty} e_\tau e_{\tau+m}\sigma^2.$$

By direct computation, we have

$$\mathrm{Cov}_m(y_t) = \sum_{l=1}^{p} \tilde{\zeta}_{lm} s_{l0} \lambda_{lm^\star}\sigma^2, \tag{A.2}$$

with

$$s_{l0} = \sum_{r=1}^{p} \frac{\tilde{\zeta}_{r0}}{(1-\phi_l\phi_r)},$$

and

$$\lambda_{lm} = \sum_{k=0}^{q}\theta_k^2 + \sum_{d=1}^{m^\star}\sum_{\kappa=0}^{q-d}\theta_\kappa\theta_{\kappa+d}(\phi_l^d+\phi_l^{-d}) + \sum_{d=m^\star+1}^{q}\sum_{\kappa=0}^{q-d}\theta_\kappa\theta_{\kappa+d}(\phi_l^d+\phi_l^{d-2m}),$$

where $m^\star = \min(m,q)$ and $\tilde{\zeta}_{lm}$ is given by equation (A.1).

It is readily verified that

$$s_{l0} = \frac{1}{\prod_{r=1}^{p}(1-\phi_l\phi_r)}.$$

Therefore, we may write

$$\mathrm{Cov}_m(y_t) = \sum_{l=1}^{p} \zeta_{lm}\lambda_{lm^\star}\sigma^2, \tag{A.3}$$

with

$$\zeta_{lm} = \frac{\phi_l^{p-1+m}}{\prod_{r=1}^{p}(1-\phi_l\phi_r)\prod_{\substack{r=1\\ r\neq l}}^{p}(\phi_l-\phi_r)},$$

where λ_{lm} is given by equation (A.2).

■

PROOF OF PROPOSITION 2

From equation (3) and the results in Proposition 1, we have

$$\mathrm{Cov}_m(y_t) = \sum_{i=1}^{n}\mathrm{Cov}_m(y_{it}) = \sum_{i=1}^{n}\sum_{l=1}^{p}\zeta_{lm}\lambda_{il,m_i^\star}\sigma_{ii},$$

where

$$\lambda_{il,m} = \sum_{k=0}^{q_i} \theta_{ik}^2 + \sum_{d=1}^{m_i^\star} \sum_{\kappa=0}^{q_i-d} \theta_{i\kappa}\theta_{i\kappa+d}(\phi_l^d + \phi_l^{-d}) + \sum_{d=m_i^\star+1}^{q_i} \sum_{\kappa=0}^{q_i-d} \theta_{i\kappa}\theta_{i\kappa+d}(\phi_l^d + \phi_l^{d-2m}),$$

and ζ_{lm} is given by equation (A.3).

■

PROOF OF PROPOSITION 3

The irf of the ARMA(p, q_i) process y_{it} in equation (4) is given by

$$y_{it} = \mu_i + \sum_{\tau=0}^{\infty} e_{i\tau}\epsilon_{it-\tau},$$

with

$$e_{i\tau} = \sum_{l=1}^{p} \sum_{\kappa=0}^{\min(\tau,q_i)} \tilde{\zeta}_{l,\tau-\kappa}\theta_{i\kappa},$$

where $\tilde{\zeta}_{lm}$ is given by equation (A.1).

Multiplying $(y_{it} - \mu_i)$ by $(y_{j,t-m} - \mu_j)$ and taking expected values, under the assumption that the elements of $\mathbf{\Sigma}$ on the off-diagonal positions are not zero, gives the cross-covariances between y_{it} and y_{jt}, at lag m $(m \in \mathbb{N})$:

$$\text{Cov}(y_{it}, y_{jt-m}) = \sum_{\tau=0}^{\infty} e_{j\tau}e_{i,\tau+m}\sigma_{ij}. \tag{A.4}$$

Straightforward algebra establishes that

$$\text{Cov}(y_{it}, y_{jt-m}) = \sum_{l=1}^{p} \tilde{\zeta}_{lm} s_{l0} \lambda_{ij,lm} \sigma_{ij},$$

with

$$\lambda_{ij,lm} = \sum_{c=0}^{q_j} \sum_{d=0}^{q_i'} \theta_{id}\theta_{j,d+c}\phi_l^c + \sum_{c=1}^{m_i^\star} \sum_{d=0}^{q_j'} \theta_{jd}\theta_{i,d+c}\phi_l^{-c} + \sum_{c=m_i^\star+1}^{q_i} \sum_{d=0}^{q_j'} \theta_{jd}\theta_{i,d+c}\phi_l^{c-2m},$$

where s_{l0} is given by equation (A.2).

Recall that $s_{l0} = \frac{1}{\prod_{r=1}^{p}(1-\phi_l\phi_r)}$. Then we may write

$$\mathrm{Cov}(y_{it}, y_{jt-m}) = \sum_{l=1}^{p} \zeta_{lm}\lambda_{ij,lm}\sigma_{ij},$$

where ζ_{lm} is given by equation (A.3).

Using the preceding equation, equations (3) and (4), we obtain the lag-m acv of y_t :

$$\mathrm{Cov}_m(y_t) = \sum_{j=1}^{n}\sum_{i=1}^{n} \mathrm{Cov}(y_{it}, y_{jt-m}) = \sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{l=1}^{p} \zeta_{lm}\lambda_{ij,lm}\sigma_{ij}.$$

■

Appendix B

PROOF OF THEOREM 1

The irf of the ARMA(p_i, q_i) process y_{it} in equation (8) is given by

$$y_{it} = \mu_i + \sum_{\tau=0}^{\infty} e_{i\tau}\epsilon_{it-\tau}, \tag{B.1}$$

where

$$e_{i\tau} = \sum_{l=1}^{p_i}\sum_{\kappa=0}^{\min(\tau,q_i)} \zeta_{il,\tau-\kappa}\theta_{i\kappa}, \tag{B.1a}$$

and

$$\zeta_{il,m} = \frac{\phi_{il}^{p_i-1+m}}{\prod_{\substack{r=1 \\ r\neq l}}^{p_i}(\phi_{il}-\phi_{ir})}.$$

The same argument leading to equation (A.4) can be used to establish that

$$\mathrm{Cov}(y_{it}, y_{j,t-m}) = \sum_{\tau=0}^{\infty} e_{j\tau}e_{i,\tau+m}\sigma_{ij}.$$

Using equation (B.1a) this can be expressed as

$$\mathrm{Cov}(y_{it}, y_{j,t-m}) = (\sum_{l=1}^{p_i} \zeta_{il,m}s_{j0,il}\lambda_{ij,lm} + \sum_{r=1}^{p_j} \zeta_{jr,m}s_{i0,jr}\lambda_{rm,ij})\sigma_{ij}, \tag{B.2}$$

with

$$s_{j0,il} = \sum_{r=1}^{p_j} \frac{\zeta_{jr,0}}{(1-\phi_{il}\phi_{jr})},$$

and

$$\lambda_{ij,lm} = \sum_{c=0}^{q_j} \sum_{d=0}^{q'_i} \theta_{id}\theta_{j,d+c}\phi_{il}^{c} + \sum_{c=1}^{m_i^\star} \sum_{d=0}^{q'_j} \theta_{jd}\theta_{i,d+c}\phi_{il}^{-c},$$

$$\lambda_{rm,ij} = \sum_{c=m_i^\star+1}^{q_i} \sum_{d=0}^{q'_j} \theta_{jd}\theta_{id+c}\phi_{jr}^{c-2m},$$

where $q'_i = min(q_i, q_j - c)$, $q'_j = min(q_j, q_i - c)$, $m_i^\star = min(m, q_i)$ and $\zeta_{il,m}$ is given by equation (B.1a). Further, it can be shown that

$$s_{j0,il} = \frac{1}{\prod_{r=1}^{p_j}(1-\phi_{il}\phi_{jr})}.$$

Therefore, equation (B.2) becomes

$$\mathrm{Cov}(y_{it}, y_{j,t-m}) = (\sum_{l=1}^{p_i} \zeta_{il,jm}\lambda_{ij,lm} + \sum_{r=1}^{p_j} \zeta_{jr,im}\lambda_{rm,ij})\sigma_{ij},$$

where

$$\zeta_{il,jm} = \frac{\zeta_{il,m}}{\prod_{r=1}^{p_j}(1-\phi_{il}\phi_{jr})}.$$

Thus, the acv of y_t, at lag m, is given by

$$\mathrm{Cov}_m(y_t) = \sum_{j=1}^{n}\sum_{i=1}^{n} \mathrm{Cov}(y_{it}, y_{j,t-m}) = \sum_{j=1}^{n}\sum_{i=1}^{n}(\sum_{l=1}^{p_i} \zeta_{il,jm}\lambda_{ij,lm} + \sum_{r=1}^{p_j} \zeta_{jr,im}\lambda_{rm,ij})\sigma_{ij}.$$

■

References

[1] Anderson, O. D. (1976). Time Series Analysis and Forecasting: the Box-Jenkins Approach. Butterworths, London.

[2] Anderson, T. W. (1971). The Statistical Analysis of Time Series. Wiley, New York.

[3] Baillie, R. T., and T. Bollerslev (1990). A Multivariate Generalized ARCH Approach to Modeling Risk Premia in Forward Foreign Exchange Rate Markets. Journal of International Money and Finance, 16, 109-124.

[4] Baillie, R. T., and H. Chung (2001). Estimation of GARCH Models from the Autocorrelations of the Squares of a Process. Journal of Time Series Analysis, *6*, 631-650.

[5] Baillie, R. T., and R. J. Myers (1991). Bivariate GARCH Estimation of Optimal Commodity Futures Hedge. Journal of Applied Econometrics, 16, 109-124.

[6] Bollerslev, T., Engle, R. F., and J. M. Wooldridge (1988). A Capital Asset Pricing Model with Time Varying Covariances. Journal of Political Economy, 96, 116-131.

[7] Box, G. E. P., and G. M. Jenkins (1970). Time Series Analysis Forecasting and Control. Holden Day, San Francisco (Rev. Ed. 1976).

[8] Brockwell, P. J., and R. A. Davis (1987). Time Series: Theory and Methods. Springer-Verlag, New York.

[9] Gourieroux, C., and A. Monfort (1997). Time Series and Dynamic Models. Cambridge University Press, Cambridge.

[10] Granger, C. W. J., and M. J. Morris (1976). Time Series Modelling and Interpretation. Journal of Royal Statistical Society, 139, 246-257.

[11] Granger, C. W. J., and P. Newbold (1986). Forecasting Economic Time Series. Academic Press, London (2nd Ed.).

[12] Grier, K., and M. Perry (2000). The Effects of Real and Nominal Uncertainty on Inflation and Output Growth: Some GARCH-M Evidence. Journal of Applied Econometrics, 15, 45-58.

[13] Hamilton, J. D. (1994). Time Series Analysis. Princeton University Press, Princeton.

[14] Hannan, E. J. (1967). Time Series Analysis. Science Paperbacks, London.

[15] Hannan, E. J. (1970). Multiple Time Series. Wiley, New York.

[16] Jenkins, G. M., and D. G. Watts (1968). Spectral Analysis and its Applications. Holden-Day, San Francisco.

[17] Karanasos, M. (1998). A New Method for Obtaining the Autocovariance of an ARMA Model: an Exact Form Solution. Econometric Theory, 14, 622-640.

[18] Karanasos, M. (2001). Prediction in ARMA Models with GARCH in Mean Effects. Journal of Time Series Analysis, 22(5), 555-576.

[19] Karolyi, A. (1995). A Multivariate GARCH Model of International Transmission of Stock Returns and Volatility: the Case of United States and Canada. Journal of Business and Economic Statistics, 13, 11-25.

[20] Kroner, K. F., and S. Claessens (1991). Optimal Currency Composition of External Debt: Applications to Indonesia and Turkey. Journal of International Money and Finance, 10, 131-148.

[21] Lutkepohl, H. (1993). Introduction to Multiple Time Series Analysis. Springer-Verlag, Berlin.

[22] McCurby, T. H., and I. G. Morgan (1991). Tests for Systematic Risk Components in Deviations from Uncovered Interest Rate Parity. Review of Economic Studies, 58, 587-602.

[23] Mc Leod, I. (1975). Derivation of the Theoretical Autocovariance Function of Autoregressive-moving average Time Series. Applied Statistics, 24, 255-256. Correction, 26 (1977), 194.

[24] Miller, K. S. (1968). Linear Difference Equations. W. A. Benjamin, New York.

[25] Mittnik, S. (1987). Non-recursive Methods for Computing the Coefficients of the Autoregressive and the Moving-average Representation of Mixed ARMA Processes. Economics Letters, 23, 279-284.

[26] Nerlove, M., Grether, D. M., and J. L. Carvalho (1979). Analysis of Economic Time Series: a Synthesis. Academic Press, New York.

[27] Ng, L. (1991). Tests of the CAPM with Time Varying Covariances: a Multivariate GARCH Approach. Journal of Finance, 46, 1507-1521.

[28] Pagan, A. (1996). The Econometrics of Financial Markets. Journal of Empirical Finance, 3, 15-102.

[29] Pandit, S. M. (1973). Data Dependent Systems: Modeling Analysis and Optimal Control via Time Series. Ph.D. Thesis, University of Wiskonsin-Madison.

[30] Pandit, S. M., and S. M. Wu (1983). Time Series and System Analysis with Applications. Wiley, New York.

[31] Priestley, M. B. (1981). Spectral Analysis and Time Series. Academic Press, London.

[32] Pugachev, V. S. (1957). Theory of Random Functions and its Applications to Automatic Control Problems. Russian (English Trans. of 3rd Ed. by O. M. Blum, Pargamon Press, New York, 1965).

[33] Reinsel, G. C. (1993). Elements of Multivariate Time Series Analysis. Springer-Verlag, New York.

[34] Sargent, T. J. (1979). Macroeconomic Theory. Academic Press, New York (2nd Ed. 1987).

[35] Shephard, N. (1996). Statistical Aspects of ARCH and Stochastic Volatility, in: Cox, D. R., O. E. Barndorff-Nielsen and D. V. Hinkley, Eds., Statistical Models in Econometrics, Finance and other Fields, 1-67.

[36] Tunnicliffe Wilson, G. (1979). Some Efficient Computational Procedures for High Order ARMA Models. Journal of Statistical Computation and Simulation, 8, 301-309.

[37] Wei, W. S. (1989). Time Series Analysis: Univariate and Multivariate Methods. Addison Wesley, New York.

[38] Wiener, N. (1949). Extrapolation, Interpolation and Smoothing of Stationary Time Series. Wiley, New York.

[39] Wold, H. O. (1938). A Study in the Analysis of Stationary Time Series. Almquist and Wicksell, Stockholm (2nd Ed. 1954).

[40] Yule, G. U. (1921). On the Time Correlation Problem, with Especial Reference to the Variate-difference Correlation Method. Journal of Royal Statistical Society, Series A, 84, 497-526.

[41] Yule, G. U. (1927). On a Method of Investigating Periodicities in Disturbed Series, with Special Reference to Wolfer's Sunspot Numbers. Philosophical Transactions, A226, p. 267.

[42] Zinde-Walsh, V. (1988). Some Exact Formulae for Autoregressive Moving Average Processes. Econometric Theory, 4, 384-402.