ON THE AUTOCORRELATION PROPERTIES OF LONG-MEMORY GARCH PROCESSES

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Abstract. This paper derives the autocorrelation function of the squared values of long-memory GARCH processes. Such processes are of much interest as they can produce the long-memory conditional heteroskedasticity that many high-frequency financial time series exhibit. An empirical application illustrating the practical use of our results is also discussed.

Keywords. Autocorrelation function; fractionally integrated GARCH process; long-memory GARCH process.

1. Introduction

A common finding in much of the empirical literature on the second-order structure of high-frequency financial time series is that sample autocorrelations for squared or absolute-valued observations tend to decay very slowly and remain fairly large at long lags (e.g. Dacorogna *et al.*, 1993; Ding *et al.*, 1993; Bollerslev and Mikkelsen, 1996; Ding and Granger, 1996; Breidt *et al.*, 1998). As a consequence, many researchers have proposed extensions of generalized autoregressive conditionally heteroskedastic (GARCH) models which can produce such long-memory behaviour; examples include the models discussed in Robinson (1991), Ding and Granger (1996), Baillie *et al.* (1996), Bollerslev and Mikkelsen (1996), Robinson and Zaffaroni (1997), and Robinson and Henry (1999), *inter alia*.

In this paper, we focus on a class of long-memory GARCH (LMGARCH) processes that belong to the family of conditionally heteroscedastic processes introduced by Robinson (1991). These processes are very closely related to the fractionally integrated GARCH (FIGARCH) processes proposed by Baillie *et al.* (1996) and share some of the features of fractional ARIMA processes. In particular, shocks to the conditional variance of an LMGARCH process eventually die away to zero (in a forecasting sense), but shock dissipation occurs at a slow hyperbolic rate rather than the faster geometric rate that is characteristic of weakly stationary GARCH processes.

Although LMGARCH models have become increasingly popular in practice, the statistical properties of time series whose behaviour is governed by such models remain largely unexplored. The present paper is intended as a first step in closing this gap, its contribution being the derivation of convenient representations for the autocorrelation function of the squared values of LMGARCH processes. Such representations are not only of theoretical interest; the practitioner could assess the adequacy of an empirical LMGARCH model by examining whether the qualitatively important features of the correlogram of the squared observations are captured by the autocorrelation function implied by the fitted model.

The remainder of the paper proceeds as follows. Section 2 lays out the models of interest, assumptions and notation. Section 3 presents the autocorrelation functions for squared LMGARCH processes. Section 4 discusses an empirical example. Section 5 concludes.

2. LONG-MEMORY GARCH PROCESSES

To establish terminology and notation, recall from Bollerslev (1986) that a GARCH(p, q) process $\{\varepsilon_t\}$ is defined by the equations

$$\varepsilon_t^2 = h_t \xi_t^2, \tag{1}$$

$$B'(L)h_t = \omega' + A'(L)\varepsilon_t^2, \tag{2}$$

where $\{\xi_t,\ t = 0,\ \pm 1,\ \pm 2,\ldots\}$ are independent and identically distributed random variables with $\mathbb{E}(\xi_t) = \mathbb{E}(\xi_t^2 - 1) = 0$

$$A'(L) \triangleq \sum_{j=1}^{q} a'_j L^j \quad \text{and} \quad B'(L) \triangleq 1 - \sum_{j=1}^{p} \beta'_j L^j.$$

L stands for the lag operator and the symbol '$\triangleq$' is used to indicate equality by definition. It follows that $\{\varepsilon_t^2\}$ admits the ARMA(p^*, p) representation

$$A^*(L)\varepsilon_t^2 = \omega' + B'(L)v_t, \quad v_t \triangleq \varepsilon_t^2 - h_t, \tag{3}$$

where

$$A^*(L) \triangleq 1 - \sum_{j=1}^{p^*} a_j^* L^j, \quad p^* = \max\{p, q\}, \quad a_j^* = a'_j + \beta'_j \ (j = 1, \ldots, p^*),$$

and $\{v_t\}$ is, by construction, a martingale-difference sequence relative to the σ-field generated by $\{\varepsilon_s,\ s \leqslant t\}$.

The class of GARCH processes can be generalized by allowing $\{\varepsilon_t^2\}$ to satisfy the equation (cf. Robinson, 1991)

$$\varepsilon_t^2 = \omega + \Omega(L)v_t, \tag{4}$$

for some $\omega \in (0, \infty)$ and

$$\Omega(L) \triangleq \sum_{j=0}^{\infty} \omega_j L^j, \quad 0 < \sum_{j=0}^{\infty} \omega_j^2 < \infty.$$

A strictly stationary GARCH(p, q) process is a special case of (4) with the coefficients $\{\omega_j, j \geqslant 0\}$ declining towards zero geometrically fast so that $\omega_j = O(\lambda^j)$ as $j \to \infty$ for some $\lambda \in (0, 1)$. When $\mathbb{E}(\varepsilon_t^4) < \infty$, the geometric decay of $\{\omega_j, j \geqslant 0\}$ implies that the autocorrelations $\{\rho_n(\varepsilon_t^2) \triangleq \mathrm{Corr}(\varepsilon_{t+n}^2, \varepsilon_t^2), n \geqslant 1\}$ are also geometrically decaying. Hence, $\{\varepsilon_t^2\}$ exhibits short memory, in the sense that the series $\sum_{n=0}^{\infty} \rho_n(\varepsilon_t^2)$ is absolutely convergent.

The specification in (4) also includes processes for which the autocorrelations $\{\rho_n(\varepsilon_t^2), n \geqslant 1\}$ decay at a rate slower than geometric. One possibility is to allow the coefficients $\{\omega_j, j \geqslant 0\}$ to decay hyperbolically so that $\omega_j \sim Cj^{-\delta}$ as $j \to \infty$ for some $\delta \in (1, \infty)$. (Henceforth, C denotes a generic finite positive constant, not necessarily the same throughout, and $a_n \sim b_n$ as $n \to \infty$ signifies that $\lim_{n \to \infty} |a_n|/b_n = 1$). An important finite parameterization of $\Omega(L)$ that allows for such behaviour is

$$\Omega(L) = \frac{B(L)}{A(L)(1 - L)^d}, \tag{5}$$

for some $d \in (0, \frac{1}{2})$, with the lag polynomials

$$A(L) \triangleq 1 - \sum_{j=1}^{q} a_j L^j = \prod_{j=1}^{q} (1 - \alpha_j L) \quad \text{and} \quad B(L) \triangleq 1 - \sum_{j=1}^{p} \beta_j L^j$$

being such that $|A(z)| > 0$ and $|B(z)| > 0$ for all complex-valued z on the closed unit disk (see e.g. Robinson and Zaffaroni, 1997; Robinson and Henry, 1999). The fractional-difference operator $(1-L)^d$ in (5) is defined as the series

$$(1 - L)^d \triangleq F(-d, 1; 1; L) = \sum_{j=0}^{\infty} \frac{\Gamma(j - d)}{\Gamma(-d)\Gamma(j + 1)} L^j = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j L^j,$$

where

$$F(a, b; c; z) \triangleq \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}$$

is the Gaussian hypergeometric series, $(b)_j \triangleq \prod_{i=0}^{j-1} (b + i)$ is Pochhammer's symbol for the shifted factorial function (with $(b)_0 \triangleq 1$), and $\Gamma(\cdot)$ is the gamma function.

It follows from (4) and (5) that the stochastic volatility h_t obeys the equation

$$h_t = \omega\Psi(1) + [1 - \Psi(L)]\varepsilon_t^2,$$

where

$$\Psi(L) \triangleq 1 - \sum_{j=1}^{\infty} \psi_j L^j = \frac{A(L)(1 - L)^d}{B(L)},$$

with $\psi_j \geqslant 0$ ($j \geqslant 1$). Furthermore, as

$$\mathrm{Var}(\varepsilon_t^2) = \mathbb{E}(v_t^2)\sum_{j=0}^{\infty}\omega_j^2 \quad \text{and} \quad \mathbb{E}(v_t^2) = \{1 - [1/\mathbb{E}(\xi_t^4)]\}\mathbb{E}(\varepsilon_t^4)$$

under (4), we have

$$\mathbb{E}(\varepsilon_t^4) = \frac{\omega^2}{1 - \{1 - [1/\mathbb{E}(\xi_t^4)]\}\sum_{j=0}^{\infty}\omega_j^2}.$$

Hence, square integrability of $\{\varepsilon_t^2\}$ requires that $0 < \mathbb{E}(\xi_t^4) < \infty$ and

$$\left\{1 - \frac{1}{\mathbb{E}(\xi_t^4)}\right\}\sum_{j=0}^{\infty}\omega_j^2 < 1. \tag{6}$$

When, for example, ξ_t is normally distributed, this condition becomes $0 < \sum_{j=0}^{\infty}\omega_j^2 < 3/2$ and we have

$$\mathbb{E}(\varepsilon_t^4) = \frac{\omega^2}{1 - \frac{2}{3}\sum_{j=0}^{\infty}\omega_j^2} < \infty.$$

Under (4)–(5), the coefficients $\{\omega_j, j \geqslant 0\}$ decay at a slow hyperbolic rate so that $\omega_j \sim Cj^{d-1}$ as $j \to \infty$. This in turn implies that the autocorrelations $\{\rho_n(\varepsilon_t^2), n \geqslant 1\}$ satisfy

$$\rho_n(\varepsilon_t^2) = \frac{\sum_{j=0}^{\infty}\omega_j\omega_{j+n}}{\sum_{j=0}^{\infty}\omega_j^2} \sim Cn^{2d-1} \quad \text{as } n \to \infty, \tag{7}$$

provided $\mathbb{E}(\varepsilon_t^4) < \infty$. Hence, when the fourth moment of the ε_t exists, $\{\varepsilon_t^2\}$ is a weakly stationary process which exhibits long memory for all $d \in (0, \frac{1}{2})$, in the sense that the series $\sum_{n=0}^{\infty}|\rho_n(\varepsilon_t^2)|$ is properly divergent. For this reason, we shall refer to a process $\{\varepsilon_t\}$ satisfying (4) and (5) as an LMGARCH(p, d, q) process.

A model closely related to the LMGARCH(p, d, q) specification in (4)–(5) was considered by Baillie *et al*. (1996), who defined a FIGARCH(p, d, q) process via the equation

$$A(L)(1 - L)^d\varepsilon_t^2 = \omega + B(L)v_t. \tag{8}$$

The FIGARCH(p, d, q) process is strictly stationary and ergodic but not square integrable (see Zaffaroni, 2000). However, as the 'autocorrelations' $(\sum_{j=0}^{\infty}\omega_j\omega_{j+n})/(\sum_{j=0}^{\infty}\omega_j^2)$ in (7) are well defined even if $\mathbb{E}(\varepsilon_t^4) = \infty$ (cf. Henry, 2001), it is not difficult to show that the FIGARCH(p, d, q) and LMGARCH(p, d, q) processes have the same second-order structure when condition (6) is satisfied.

Finally, it is worth mentioning that Giraitis *et al*. (2000) have recently studied the properties of infinite-order ARCH processes. Their results, however, do not apply for the specification in (4)–(5) as it does not satisfy the condition $\mathbb{E}(\xi_t^2)[1 - \Psi(1)] < 1$, which was shown to be sufficient for strict stationarity of $\{\varepsilon_t^2\}$. The exact properties of processes satisfying (4)–(5) thus remain an open question.

3. AUTOCORRELATION STRUCTURE OF LMGARCH PROCESSES

In this section of the paper, we establish the autocorrelation properties of LMGARCH processes. We begin by considering some low-order processes which have proved to be useful in modelling a variety of financial time series and then proceed to examine the general LMGARCH(p, d, q) case.

3.1. *LMGARCH(1, d, 1) process*

The LMGARCH(1, d, 1) process is defined via the fractional ARIMA(1, d, 1) equation

$$\varepsilon_t^2 = \omega + (1 - \beta_1 L)(1 - \alpha_1 L)^{-1}(1 - L)^{-d} v_t. \tag{9}$$

We begin by giving the infinite moving-average representation of ε_t^2.

LEMMA 1. *The process $\{\varepsilon_t^2\}$ admits the infinite moving-average representation*

$$\varepsilon_t^2 = \omega + \sum_{j=0}^{\infty} \omega_j v_{t-j}, \tag{10}$$

where

$$\omega_j = \binom{-d}{j}(-1)^j + \sum_{k=1}^{j} \binom{-d}{j-k}(-1)^{j-k}(\alpha_1^k - \alpha_1^{k-1}\beta_1). \tag{11}$$

PROOF. As

$$(1 - \kappa L)^{-d} = \sum_{j=0}^{\infty} \binom{-d}{j}(-\kappa)^j L^j, \tag{12}$$

we have

$$\frac{1 - \beta_1 L}{1 - \alpha_1 L} = \sum_{j=0}^{\infty} \binom{-1}{j}(-\alpha_1 L)^j (1 - \beta_1 L) = 1 + \sum_{j=1}^{\infty} (\alpha_1^j - \alpha_1^{j-1}\beta_1) L^j,$$

and hence (10) follows from (9). QED

In the first proposition, we obtain the autocorrelation function of $\{\varepsilon_t^2\}$.

PROPOSITION 1. *The autocorrelation function of* $\{\varepsilon_t^2\}$ *is given by*

$$\rho_n(\varepsilon_t^2) = \gamma_n/\gamma_0, \quad n \geqslant 1, \tag{13}$$

where

$$\begin{aligned}\gamma_n =& \frac{\Gamma(1-2d)}{(1-\alpha_1^2)\Gamma(d)\Gamma(1-d)}\left\{\frac{\Gamma(d+n)}{\Gamma(1-d+n)}[(1+\beta_1^2-\beta_1\alpha_1)\right.\\ &\times F(d+n,1;1-d+n;\alpha_1) - \alpha_1\beta_1 F(d-n,1;1-d-n;\alpha_1)]\\ &+\frac{\Gamma(d+n-1)}{\Gamma(n-d)}[\alpha_1(1+\beta_1^2)-\beta_1]F(d-n+1,1;2-d-n;\alpha_1)\\ &\left.-\frac{\Gamma(d+n+1)}{\Gamma(2-d+n)}\beta_1 F(d+n+1,1;2-d+n;\alpha_1)\right\}, \quad n \geqslant 0. \end{aligned} \tag{14}$$

PROOF. On account of (10) and (11), we have

$$\rho_n(\varepsilon_t^2) = \frac{\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty}\binom{-d}{|n-j|+k}\binom{-d}{k}(-1)^{|n-j|}\sum_{i=0}^{\infty}\pi_i\pi_{i+|j|}}{\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty}\binom{-d}{|-j|+k}\binom{-d}{k}(-1)^{|-j|}\sum_{i=0}^{\infty}\pi_i\pi_{i+|j|}}, \tag{15}$$

where $\pi_0 \triangleq 1$ and $\pi_i = \alpha_1^i - \alpha_1^{i-1}\beta_1$ for $i \geqslant 1$. But as

$$\sum_{k=0}^{\infty}\binom{-d}{|n-j|+k}\binom{-d}{k}(-1)^{|n-j|} = \frac{\Gamma(1-2d)\Gamma(d+|n-j|)}{\Gamma(d)\Gamma(1-d)\Gamma(1-d+|n-j|)},$$

it follows that

$$\rho_n(\varepsilon_t^2) = \frac{\sum_{j=-\infty}^{\infty}\frac{\Gamma(d+|n-j|)}{\Gamma(1-d+|n-j|)}\{(1+\beta_1^2)\alpha_1^{|j|}-\beta_1(\alpha_1^{||j|-1|}+\alpha_1^{|j|+1})\}}{\sum_{j=-\infty}^{\infty}\frac{\Gamma(d+|-j|)}{\Gamma(1-d+|-j|)}\{(1+\beta_1^2)\alpha_1^{|j|}-\beta_1(\alpha_1^{||j|-1|}+\alpha_1^{|j|+1})\}}.$$

Finally, from the fact that

$$\sum_{j=0}^{\infty}\frac{\Gamma(d+|n-j|)}{\Gamma(1-d+|n-j|)}\alpha_1^j = \frac{\Gamma(d+n)}{\Gamma(1-d+n)}F(d-n,1;1-d-n;\alpha_1),$$

we obtain (13)–(14) by straightforward manipulation. QED

In Figure 1, we plot the theoretical autocorrelation function of a squared LMGARCH(1, d, 1) process with $\alpha_1 = 0.1$, $\beta_1 = 0.2$ and $d \in \{0.2, 0.3\}$.[1] As expected, the autocorrelations decay at a very slow rate, much slower than the geometric rate that is characteristic of weakly stationary GARCH processes.

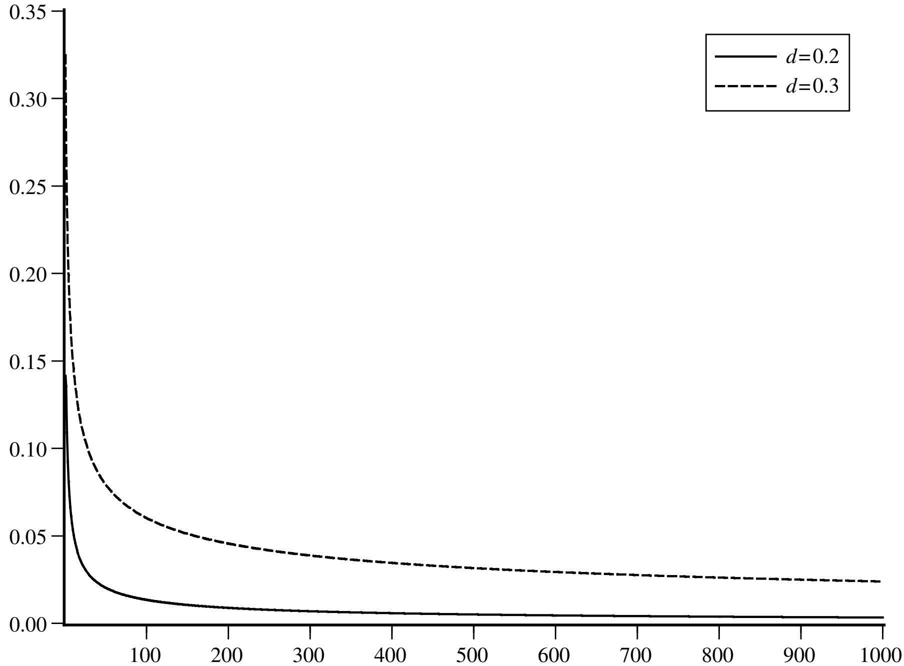

FIGURE 1. Autocorrelation function of squared LMGARCH(1, d, 1) process.

3.2. *LMGARCH(p, d, 0) process*

Now consider the LMGARCH(p, d, 0) process defined via the fractional ARIMA(0, d, p) equation

$$\varepsilon_t^2 = \omega + B(L)(1 - L)^{-d} v_t. \tag{16}$$

For the process in (16), we have the following result.

LEMMA 2. *The process $\{\varepsilon_t^2\}$ admits the infinite moving-average representation*

$$\varepsilon_t^2 = \omega + \sum_{j=0}^{\infty} \omega_j v_{t-j}, \tag{17}$$

where

$$\omega_j = \sum_{k=0}^{j} \binom{-d}{j-k} (-1)^{j-k} \pi_k, \tag{18}$$

and

$$\pi_k \triangleq \sum_{r=0}^{\min\{k,p\}} (-\beta_r) \quad (\beta_0 \triangleq -1).$$

PROOF. The desired result is obtained straightforwardly from (16) by using (12). QED

The autocorrelations of the process defined by (16) are obtained next.

PROPOSITION 2. *The autocorrelation function of* $\{\varepsilon_t^2\}$ *is given by*

$$\rho_n(\varepsilon_t^2) = \gamma_n/\gamma_0, \quad n \geqslant 1, \tag{19}$$

where

$$\gamma_n = \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \sum_{l=-p}^{p} \Phi_{|l|} \frac{\Gamma(d+n+l)}{\Gamma(1-d+n+l)}, \quad n \geqslant 0, \tag{20}$$

with

$$\Phi_{|l|} \triangleq \sum_{k=0}^{p-|l|} \beta_k \beta_{k+|l|} \quad (\beta_0 \triangleq -1).$$

PROOF. Using the fact that $\rho_n(\varepsilon_t^2) = (\sum_{j=0}^{\infty} \omega_j \omega_{j+n})/(\sum_{j=0}^{\infty} \omega_j^2)$, we have in view of (18) that

$$\rho_n(\varepsilon_t^2) = \frac{\sum_{l=0}^{p} \Phi_l H_l \left[\sum_{j=0}^{\infty} \binom{-d}{j} \binom{-d}{j+|n-l|} (-1)^{|n-l|} + \sum_{j=0}^{\infty} \binom{-d}{j} \binom{-d}{j+n+l} (-1)^{n+l} \right]}{\sum_{l=0}^{p} \Phi_l H_l \left[\sum_{j=0}^{\infty} \binom{-d}{j} \binom{-d}{j+|-l|} (-1)^{|-l|} + \sum_{j=0}^{\infty} \binom{-d}{j} \binom{-d}{j+l} (-1)^{l} \right]},$$

where

$$H_l \triangleq \begin{cases} \frac{1}{2}, & \text{if } l = 0, \\ 1, & \text{if } l \neq 0. \end{cases}$$

Hence, as

$$\sum_{k=0}^{\infty} \binom{-d}{|n-l|+k} \binom{-d}{k} (-1)^{|n-l|} = \frac{\Gamma(1-2d)\Gamma(d+|n-l|)}{\Gamma(d)\Gamma(1-d)\Gamma(1-d+|n-l|)},$$

(19)–(20) follow. QED

Figure 2 shows the theoretical autocorrelation function of a squared LMGARCH(1, d, 0) process with $\beta_1 = 0.1$ and $d \in \{0.2, 0.3\}$. The shape of the autocorrelation functions is very similar to those for the LMGARCH(1, d, 0) process, exhibiting a rate of decay much slower than geometric.

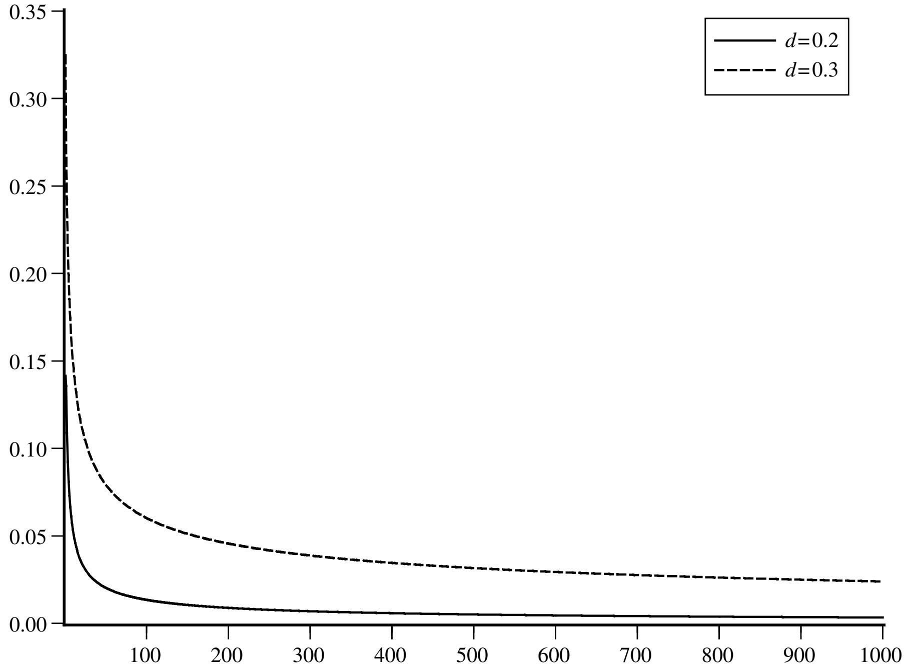

FIGURE 2. Autocorrelation function of squared LMGARCH(1, d, 0) process.

3.3. *LMGARCH(0, d, q) process*

Next consider the LMGARCH(0, d, q) process defined via the fractional ARIMA(q, d, 0) equation

$$\varepsilon_t^2 = \omega + A^{-1}(L)(1-L)^{-d}v_t, \tag{21}$$

where it is assumed that the roots of $A(z) = 0$ are simple. The moving-average representation of the process in (21) is as follows.

LEMMA 3. *The process* $\{\varepsilon_t^2\}$ *admits the infinite moving-average representation*

$$\varepsilon_t^2 = \omega + \sum_{j=0}^{\infty} \omega_j v_{t-j}, \tag{22}$$

where

$$\omega_j = \sum_{i=1}^{q} \alpha_i^+ \sum_{k=0}^{j} \binom{-d}{j-k} \alpha_i^k (-1)^{j-k}, \tag{23}$$

and

$$\alpha_i^+ \triangleq \frac{\alpha_i^{q-1}}{\prod_{k=1,k\neq i}^{q}(\alpha_i - \alpha_k)}.$$

Proof. From (21), we have that

$$\varepsilon_t^2 = \omega + (1-L)^{-d}\left[\prod_{j=1}^{q}(1-\alpha_j L)\right]^{-1} v_t.$$

Hence, in view of the fact that

$$\prod_{j=1}^{q}(1-\alpha_j L) = \sum_{i=1}^{q}\frac{\alpha_i^+}{1-\alpha_i L},$$

$$(1-L)^{-d} = \sum_{j=0}^{\infty}\binom{-d}{j}(-1)^j L^j \quad \text{and} \quad (1-\alpha_i L)^{-1} = \sum_{j=0}^{\infty}\binom{-1}{j}(-\alpha_i L)^j,$$

(22)–(23) follow. QED

The autocorrelation structure of the process defined by (21) is established next.

Proposition 3. *The autocorrelation function of* $\{\varepsilon_t^2\}$ *is given by*

$$\rho_n(\varepsilon_t^2) = \gamma_n/\gamma_0, \quad n \geqslant 1, \tag{24}$$

where

$$\gamma_n = \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\sum_{i=1}^{q}\overline{\alpha}_i\left\{\frac{\Gamma(d+n)}{\Gamma(1-d+n)}F(d+n,1;1-d+n;\alpha_i)\right.$$
$$\left. + \alpha_i\frac{\Gamma(d+n-1)}{\Gamma(n-d)}F(d-n+1,1;2-d-n;\alpha_i)\right\}, \quad n \geqslant 0, \tag{25}$$

with

$$\overline{\alpha}_i \triangleq \frac{\alpha_i^+}{\prod_{k=1}^{q}(1-\alpha_i\alpha_k)}.$$

Proof. In view of (23), we have

$$\rho_n(\varepsilon_t^2) = \frac{\sum_{j=0}^{\infty}\omega_j\omega_{j+n}}{\sum_{j=0}^{\infty}\omega_j^2} = \frac{\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty}\phi_{n-j}^k\eta_j}{\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty}\phi_{-j}^k\eta_j},$$

where

$$\phi^k_{n-j} \triangleq \binom{-d}{|n-j|+k}\binom{-d}{k}(-1)^{|n-j|} \quad \text{and} \quad \eta_j \triangleq \sum_{i=1}^{q}\sum_{r=1}^{q}\sum_{m=0}^{\infty} \alpha_i^+ \alpha_i^m \alpha_r^+ \alpha_r^{m+|j|}.$$

But as

$$\sum_{k=0}^{\infty} \phi^k_{n-j} = \frac{\Gamma(1-2d)\Gamma(d+|n-j|)}{\Gamma(d)\Gamma(1-d)\Gamma(1-d+|n-j|)} \quad \text{and} \quad \sum_{k=1}^{q} \frac{\alpha_i^+ \alpha_k^+}{1-\alpha_i\alpha_k} = \overline{\alpha}_i,$$

it follows that

$$\rho_n(\varepsilon_t^2) = \frac{\sum_{j=-\infty}^{\infty} \frac{\Gamma(d+|n-j|)}{\Gamma(1-d+|n-j|)} \sum_{i=1}^{q} \overline{\alpha}_i \alpha_i^{|j|}}{\sum_{j=-\infty}^{\infty} \frac{\Gamma(d+|-j|)}{\Gamma(1-d+|-j|)} \sum_{i=1}^{q} \overline{\alpha}_i \alpha_i^{|j|}}$$
$$= \frac{\sum_{j=0}^{\infty}\sum_{i=1}^{q} \overline{\alpha}_i \alpha_i^j \left[\frac{\Gamma(d+n+j)}{\Gamma(1-d+n+j)} + \alpha_i \frac{\Gamma(d+|n-1-j|)}{\Gamma(1-d+|n-1-j|)}\right]}{\sum_{j=0}^{\infty}\sum_{i=1}^{q} \overline{\alpha}_i \alpha_i^j \left[\frac{\Gamma(d+j)}{\Gamma(1-d+j)} + \alpha_i \frac{\Gamma(d+|-1-j|)}{\Gamma(1-d+|-1-j|)}\right]}.$$

Finally, using

$$\sum_{j=0}^{\infty} \frac{\Gamma(d+n+j)}{\Gamma(1-d+n+j)} \alpha_i^j = \frac{\Gamma(d+n)}{\Gamma(1-d+n)} F(d+n, 1; 1-d+n; \alpha_i),$$

we obtain (24)–(25). QED

Remark. Our results are limited to LMGARCH(0, d, q) processes for which $\alpha_i \neq \alpha_k$ for all $i, k \in \{1,\ldots,q\}$ such that $i \neq k$. However, as Sowell (1992) remarked, this might not be an overly restrictive requirement as, in the space of polynomials of a given order, the subset which has repeated zeros is a set with zero Lebesgue measure.

Figure 3 shows a plot of the theoretical autocorrelation function of a squared LMGARCH(0, d, 1) process with $\alpha_1 = 0.1$ and $d \in \{0.2, 0.3\}$. As before, the autocorrelations decrease extremely slowly.

3.4. *LMGARCH(p, d, q) process*

We finally consider the general LMGARCH(p, d, q) process defined via (4)–(5) with the added restriction that the roots of $A(z) = 0$ are simple. The moving-average representation of such a process is given in Lemma 4.

Lemma 4. *The process $\{\varepsilon_t^2\}$ admits the infinite moving-average representation*

$$\varepsilon_t^2 = \omega + \sum_{j=0}^{\infty} \omega_j v_{t-j}, \tag{26}$$

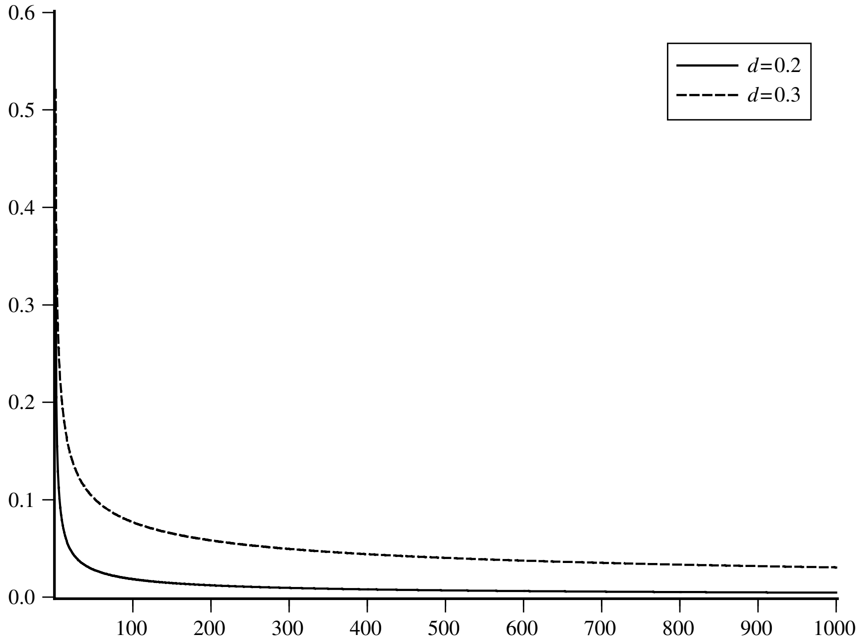

FIGURE 3. Autocorrelation function of squared LMGARCH(0, d, 1) process.

where

$$\omega_j = \sum_{i=1}^{q} \alpha_i^{+} \sum_{k=0}^{j} \binom{-d}{j-k} \pi_{ik}(-1)^{j-k}, \tag{27}$$

$$\pi_{ik} \triangleq \sum_{r=0}^{\min\{k,p\}} \alpha_i^{k-r}(-\beta_r) \;\; (\beta_0 \triangleq -1), \quad \text{and} \quad \alpha_i^{+} \triangleq \frac{\alpha_i^{q-1}}{\prod_{k=1,k\neq i}^{q}(\alpha_i - \alpha_k)}.$$

PROOF. From (4)–(5), we have that

$$\varepsilon_t^2 = \omega + (1-L)^{-d}\left[\prod_{j=1}^{q}(1-\alpha_j L)\right]^{-1} B(L)v_t.$$

Hence, on account of

$$\prod_{j=1}^{q}(1-\alpha_j L) = \sum_{i=1}^{q} \frac{\alpha_i^{+}}{1-\alpha_i L},$$

$$(1-L)^{-d}=\sum_{j=0}^{\infty}\binom{-d}{j}(-1)^jL^j \quad \text{and} \quad (1-\alpha_iL)^{-1}B(L)=\sum_{r=0}^{\infty}\pi_{ir}L^r,$$

(26)–(27) follow. QED

Next, we establish a representation for the autocorrelation function of the squared values of the LMGARCH(p, d, q) process.

THEOREM 1. *The autocorrelation function of* $\{\varepsilon_t^2\}$ *is given by*

$$\rho_n(\varepsilon_t^2)=\gamma_n/\gamma_0, \quad n\geqslant 1, \tag{28}$$

where

$$\gamma_n=\sum_{i=1}^{q}\sum_{l=0}^{p}\Phi_l\bar{\alpha}_i\Theta(d,n,l,\alpha_i), \quad n\geqslant 0, \tag{29}$$

$$\Phi_l\triangleq\sum_{k=0}^{p-l}\beta_k\beta_{k+l}\ \ (\beta_0\triangleq -1), \quad \bar{\alpha}_i\triangleq\frac{\alpha_i^+}{\prod_{k=1}^{q}(1-\alpha_i\alpha_k)},$$

$$\begin{aligned}\Theta(d,n,l,\alpha_i)\triangleq{}&\frac{\Gamma(d+n+l)}{\Gamma(1-d+n+l)}F(d+n+l,1;1-d+n+l;\alpha_i)\\&+\mathbf{1}_l\frac{\Gamma(d+n-l)}{\Gamma(1-d+n-l)}F(d-n+l,1;1-d-n+l;\alpha_i)\\&+\alpha_i\left[\mathbf{1}_l\frac{\Gamma(d+n+1-l)}{\Gamma(2-d+n-l)}F(d+n+1-l,1;2-d+n-l;\alpha_i)\right.\\&\left.+\frac{\Gamma(d+n-1+l)}{\Gamma(n-d+l)}F(d-n+1-l,1;2-d-n-l;\alpha_i)\right],\end{aligned}$$

and

$$\mathbf{1}_l\triangleq\begin{cases}0, & \text{if } l=0,\\ 1, & \text{if } l\neq 0.\end{cases}$$

PROOF. In view of (27), we have

$$\rho_n(\varepsilon_t^2)=\frac{\sum_{j=0}^{\infty}\omega_j\omega_{j+n}}{\sum_{j=0}^{\infty}\omega_j^2}=\frac{\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty}\phi_{n-j}^k\eta_j}{\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty}\phi_{-j}^k\eta_j},$$

where

$$\phi_{n-j}^k\triangleq\binom{-d}{|n-j|+k}\binom{-d}{k}(-1)^{|n-j|} \quad \text{and} \quad \eta_j\triangleq\sum_{i=1}^{q}\sum_{s=1}^{q}\sum_{m=0}^{\infty}\alpha_i^+\ \alpha_s^+\pi_{im}\pi_{s,m+|j|}.$$

But as

$$\sum_{k=0}^{\infty}\phi_{n-j}^{k}=\frac{\Gamma(1-2d)\Gamma(d+|n-j|)}{\Gamma(d)\Gamma(1-d)\Gamma(1-d+|n-j|)}\quad\text{and}\quad\sum_{k=1}^{q}\frac{\alpha_i^{+}\alpha_k^{+}}{1-\alpha_i\alpha_k}=\overline{\alpha}_i,$$

it follows that

$$\begin{aligned}\rho_n(\varepsilon_t^2)=&\frac{\sum_{j=-\infty}^{\infty}\frac{\Gamma(d+|n-j|)}{\Gamma(1-d+|n-j|)}\sum_{i=1}^{q}\sum_{l=0}^{p}\Phi_l\overline{\alpha}_i(\mathbf{1}_l\alpha_i^{||j|-l|}+\alpha_i^{|j|+l})}{\sum_{j=-\infty}^{\infty}\frac{\Gamma(d+|-j|)}{\Gamma(1-d+|-j|)}\sum_{i=1}^{q}\sum_{l=0}^{p}\Phi_l\overline{\alpha}_i(\mathbf{1}_l\alpha_i^{||j|-l|}+\alpha_i^{|j|+l})}\\
=&\left\{\sum_{l=0}^{p}\sum_{j=0}^{\infty}\Phi_l\sum_{i=1}^{q}\overline{\alpha}_i\left[\frac{\mathbf{1}_l\Gamma(d+|n-l-j|)}{\Gamma(1-d+|n-l-j|)}+\frac{\Gamma(d+n+l+j)}{\Gamma(1-d+n+l+j)}\right.\right.\\
&\left.\left.+\left(\frac{\Gamma(d+|n-1+l-j|)}{\Gamma(1-d+|n-1+l-j|)}+\frac{\mathbf{1}_l\Gamma(d+|n+1-l+j|)}{\Gamma(1-d+|n+1-l+j|)}\right)\alpha_i\right]\alpha_i^j\right\}\\
&\times\left\{\sum_{l=0}^{p}\sum_{j=0}^{\infty}\Phi_l\sum_{i=1}^{q}\overline{\alpha}_i\left[\frac{\mathbf{1}_l\Gamma(d+|-l-j|)}{\Gamma(1-d+|-l-j|)}+\frac{\Gamma(d+l+j)}{\Gamma(1-d+l+j)}\right.\right.\\
&\left.\left.+\left(\frac{\Gamma(d+|-1+l-j|)}{\Gamma(1-d+|-1+l-j|)}+\frac{\mathbf{1}_l\Gamma(d+|1-l+j|)}{\Gamma(1-d+|1+j-l|)}\right)\alpha_i\right]\alpha_i^j\right\}^{-1}.\end{aligned}$$

Hence, upon observing that

$$\sum_{j=0}^{\infty}\frac{\Gamma(d+n+l+j)}{\Gamma(1-d+n+l+j)}\alpha_i^j=\frac{\Gamma(d+n+l)}{\Gamma(1-d+n+l)}F(d+n+l,1;1-d+n+l;\alpha_i),$$

(28)–(29) are obtained. QED

4. AN EMPIRICAL ILLUSTRATION

As an empirical illustration, we examine the properties of continuously compounded daily rates of return for the Deutschmark exchange rate vis-à-vis the US Dollar over the period from 31/10/1983 to 31/12/1992 (2,394 observations in total). To evaluate how well the second-order properties of the squared exchange rate returns are approximated by various models, we compared the sample autocorrelation function of the squared returns with the theoretical autocorrelation functions of LMGARCH(1, *d*, 0), LMGARCH(0, *d*, 1), LMGARCH(1, *d*, 1), and IGARCH(1, 1) processes. The theoretical autocorrelations were evaluated using the formulae given in Propositions 1–3 and the quasi-maximum likelihood parameter estimates reported in Table I (obtained under the assumption of conditional Gaussianity). In the IGARCH(1, 1) case, we used the formula for the approximate autocorrelation function given in Ding and Granger (1996).

Figure 4 plots the theoretical and sample autocorrelations for lags up to 2392.[2] Evidently, the LMGARCH(1, *d*, 0) and LMGARCH(0, *d*, 1) processes give the

TABLE I
QUASI-MAXIMUM LIKELIHOOD ESTIMATES

	(1, d, 0)	(0, d, 1)	(1, d, 1)
d	0.2326 (0.0365)	0.1847 (0.0237)	0.3805 (0.0680)
α_1	–	−0.1260 (0.0306)	0.2742 (0.0471)
β_1	0.1973 (0.0460)	–	0.6114 (0.0620)

Notes: Figures in parentheses are asymptotic standard error values.

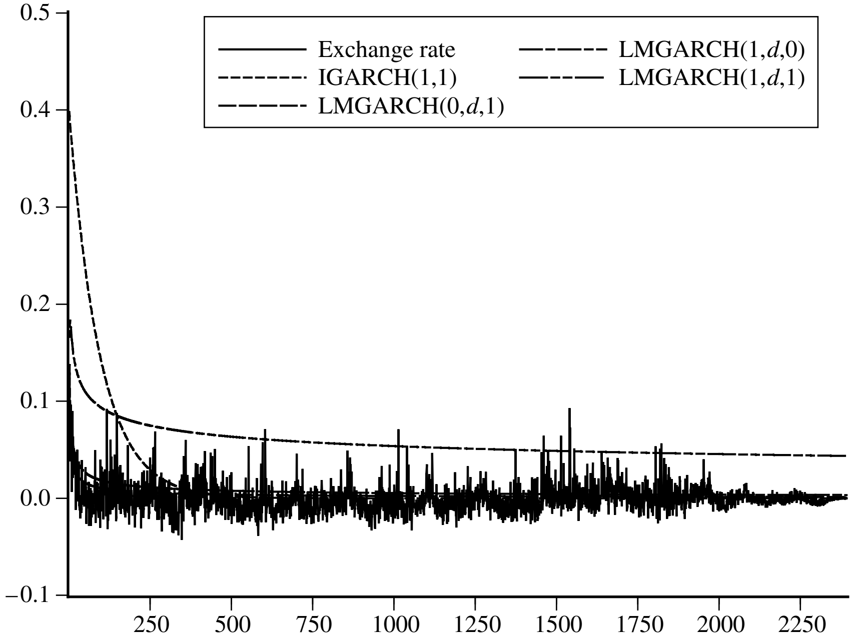

FIGURE 4. Sample autocorrelations of squared exchange-rate returns and theoretical LMGARCH autocorrelation functions.

best approximation to the autocorrelations of the squared exchange rate returns. The LMGARCH(1, d, 1) process has autocorrelations considerable larger than the sample autocorrelations of the returns. The IGARCH(1, 1) process, on the other hand, provides a very bad approximation to the first 295 sample autocorrelations of the returns but is almost as good as the LMGARCH(1, d, 0) and LMGARCH(0, d, 1) from then onwards.

5. CONCLUSION

In this paper, we have examined the dependence structure of long-memory autoregressive conditionally heteroscedastic processes. More specifically, we have obtained characterizations of the theoretical autocorrelation function of the squared values of LMGARCH processes. Such processes have been found to describe well the observed autocorrelation structure of many real-life financial time series and are thus of much interest. Using our results, one can establish what a fitted model implies about the second-order structure of the squared observations and the extent to which these characteristics are consistent with the correlogram of the data. This is illustrated in an empirical application involving foreign exchange rate data.

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NOTES

1. The Gaussian hypergeometric series was evaluated using Mathematica.
2. The quasi-maximum likelihood estimates in Table I are such that $0 < \sum_{j=0}^{\infty} \omega_j^2 < \frac{3}{2}$ for all three LMGARCH models.

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