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The second moment and the autocovariance function of the squared errors of the GARCH model

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Abstract

Since Bollerslev and Taylor indepedently introduced the GARCH model almost a decade ago many questions have remained unanswered. This paper addresses two of them. First, 'What is the autocovariance structure of the squared errors?' and second, 'What is the condition on the parameters of the GARCH (p, q) model in order for the fourth moment of the errors to exist?'. In Section 2 of this paper we answer the first question and in Section 3 we answer the second one. In a recent paper Ding and Granger introduced an extension of the GARCH(1, 1) model which they called the N-component GARCH(1, 1) model and they mentioned that it can be expressed as a GARCH(n, n) model. This GARCH(n, n) representation is presented in Section 3. Finally, in Section 3, we introduce the two component GARCH(n, n) model and we express it as a GARCH$(2n, 2n)$ model. © 1999 Elsevier Science S.A. All rights reserved.

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1. Introduction

Two of the most common empirical findings in the finance literature are that the distributions of asset returns display tails heavier than those of the normal distribution and the squared returns are highly serially correlated. The aforementioned stylized empirical regularities led some econometricians to develop models which can accommodate and account for these phenomena. Engle (1982) introduced the Autoregressive Conditional Heteroscedasticity

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(ARCH) model and Bollerslev (1986) (hereafter B), and Taylor (1986), generalized the ARCH to the GARCH model:

$$\varepsilon_{t/t-1} \sim N(0, h_t), \quad h_t = \sum_{i=1}^{p} \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_t - j.$$

This paper presents exact formulae for the second moments of the squared errors of the GARCH(p, q) model. B (1988), Ding and Granger (1996) (hereafter DG) and Karanasos (1996) (hereafter K) give the autocorrelation function of the squared errors for the GARCH(1, 1) process. In Section 2 we extend these results to the GARCH(p, q) model.

Moreover, Milhøl (1985) in Theorem 3 of his paper, gives a necessary and sufficient condition for the existence of the second-order moment of the squared errors of the ARCH(p) model. B (1986) in Theorem 2 of his paper gives a necessary and sufficient condition for the existence of the fourth moment of the errors of the GARCH(1, 1), GARCH(1, 2) and GARCH(2, 1) models. In Section 3 of this paper we present a method for obtaining the unconditional fourth moment, and hence the kurtosis coefficient of the errors of the GARCH(p, q) model.

Furthermore, several empirical studies have pointed out the long memory property (significant positive autocorrelation for many lags) of speculative returns (e.g. Ding et al., 1993; DG, 1996). Motivated by this empirical result, DG (1996) considered a new version of the GARCH(1, 1) model which they called the N-component GARCH(1, 1) model. In this model the conditional variance of the errors (h_t) is a weighted sum of N components h_{it} ($i = 1, \ldots, N$) with w_i ($i = 1, \ldots, N$) as weights, respectively. Each component is a GARCH(1, 1)-type specification. We use the exact form solutions of Sections 2 and 3 to analyze the N-component GARCH(1, 1) and the two-component GARCH(n, n) models. This is accomplished (i) by showing that the N-component GARCH(1, 1) process corresponds to a GARCH(n, n) model, and that the two-component GARCH(n, n) process is represented by a GARCH($2n, 2n$) model (Lemmas 3.1 and 3.2, respectively), and (ii) by applying the results for the general GARCH(p, q) model to obtain the second moments of the squared errors of the N-component GARCH(1, 1) and the two-component GARCH(n, n) models. Finally, Section 4 concludes.

2. Autocovariance of the GARCH model

Let ε_t^2 follow a GARCH(p, q) process, given by

$$h_t = a_0 + \sum_{i=1}^{p} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i} \Rightarrow \varepsilon_t^2$$

$$= a_0 + \sum_{i=1}^{\max(p,q)} a_i' \varepsilon_{t-i}^2 + v_t - \sum_{i=1}^{q} \beta_i v_{t-i}$$

$$\Rightarrow \prod_{i=1}^{\max(p,q)} (1 - a_i^* L)\varepsilon_t^2 = a_0 + v_t - \sum_{i=1}^{q} \beta_i v_{t-i},$$

where

$$a_1^* \neq a_j^* \quad \text{for } i \neq j,$$

$$a_i' = a_i + \beta_i \quad \text{for } i \leqslant p, q, \; a_i' = a_i \quad \text{for } i, p > q \quad \text{and} \quad a_i' = \beta_i \quad \text{for } i, q > p. \tag{2.1}$$

Theorem 2.1. The autocovariance function of the squared errors of the above GARCH(p, q) process is given by

$$\gamma_j = \begin{cases} \sum_{i=1}^{\max(p,q)} e_{ij} z_{ij} \,(2/3) f_2, & j \leqslant q-1, \\ \sum_{i=1}^{\max(p,q)} e_{ij} z_{iq} \,(2/3) f_2, & j \geqslant q, \end{cases} \tag{2.2}$$

where

$$e_{ij} = \frac{(a_i^*)^{j+\max(p,q)-1}}{\prod_{l=1}^{\max(p,q)} (1 - a_i^* a_l^*) \prod_{k=1, k\neq i}^{\max(p,q)} (a_i^* - a_k^*)}, \quad f_2 = E(\varepsilon_t^4), \tag{2.2a}$$

and

$$z_{ij} = \sum_{k=0}^{q} \beta_k^2 + \sum_{l=1}^{j} \sum_{k=0}^{q-l} \beta_k \beta_{k+l} [(a_i^*)^l + (a_i^*)^{-l}]$$
$$+ \sum_{l=j+1}^{q} \sum_{k=0}^{q-l} \beta_k \beta_{k+l} [(a_i^*)^l + (a_i^*)^{l-2j}], \; \beta_0 = -1. \tag{2.2b}$$

Observe that, for $j = q$, the third term on the right-hand side of the above equation disappears since the lower limit exceeds the upper limit of the double summation and that the autocovariance function depends on the fourth moment of the errors which we give in Section 3. □

3. Fourth moment of the GARCH model

Let h_t follow a GARCH(p, q) process, given by

$$h_t = a_0 + \sum_{i=1}^{p} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i}. \tag{3.1}$$

Theorem 3.1. The fourth moment of the errors of the above GARCH(p, q) process is given by $f_2 = 3a_0 f_1 \phi_1/(1-\delta_1)$, *where* f_1, ϕ_1 *and* δ_1 *are given by*

$$f_1 = \frac{a_0}{1-\sum_{i=1}^{p} a_i - \sum_{i=1}^{q}\beta_i}, \tag{3.2}$$

$$\begin{aligned}\phi_1 = {} & 1 + a_p + \beta_q + \sum_{j=1}^{p-1}[a_j + a_p(a_{p-j}+\beta_{p-j})]\sum_{i=1}^{p+q-2}\Gamma_{j+q-1,i} \\ & + \sum_{j=1}^{q-1}[\beta_j + \beta_q(a_{q-j}+\beta_{q-j})]\sum_{i=1}^{p+q-2}\Gamma_{ji} \\ & + \beta_q\sum_{j=1}^{p-q} a_{q+j}\sum_{i=1}^{p+q-2}\Gamma_{j+q-1,i} + a_p\sum_{j=1}^{q-p}\beta_{p+j}\sum_{j=1}^{q+p-2}\Gamma_{ji},\end{aligned} \tag{3.2a}$$

$$\begin{aligned}\delta_1 = {} & 3a_p^2 + \beta_q^2 + a_q\beta_q + a_p\beta_p + \sum_{j=1}^{p-1}[a_j + a_p(a_{p-j}+\beta_{p-j})] \\ & \times \sum_{i=1}^{p+q-2}(\gamma_i a_{i-\delta_i} + \beta_{i-\delta_i})\Gamma_{j+q-1,i} + \sum_{j=1}^{q-1}[\beta_j + \beta_q(a_{q-j}+\beta_{q-j})] \\ & \times \sum_{i=1}^{p+q-2}(\gamma_i a_{i-\delta_i} + \beta_{i-\delta_i})\Gamma_{ji} + \beta_q\sum_{j=1}^{p-q} a_{q+j}\sum_{i=1}^{p+q-2}(\gamma_i a_{i-\delta_i} + \beta_{i-\delta_i})\Gamma_{j+q-1,i} \\ & + a_p\sum_{j=1}^{q-p}\beta_{p+j}\sum_{i=1}^{p+q-2}(\gamma_i a_{i-\delta_i} + \beta_{i-\delta_i})\Gamma_{ji},\end{aligned} \tag{3.2b}$$

where Γ_{ij} *is the ijth element of the matrix* $\Gamma = A^{-1}$. A *is a* $(p+q-2)\times(p+q-2)$ *matrix and consists of four submatrices*:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

A_1 *is a* $(q-1)\times(q-1)$ *matrix. Its ijth element is* $-(\beta_{i-j} + a_{i-j} + \beta_{i+j})$, *where* $\beta_k = 0$ *for* $k > q$ *or* $k < 0$, $\beta_0 = -1$ *and* $a_k = 0$ *for* $k > p$ *or* $k \leqslant 0$. A_2 *is a* $(q-1)\times(p-1)$ *matrix. Its ijth element is* $-a_{i+j}$, *where* $a_k = 0$ *for* $k > p$. A_3 *is a* $(p-1)\times(q-1)$ *matrix. Its ijth element is* $-\beta_{i+j}$, *where* $\beta_k = 0$ *for* $k > q$. A_4 *is a* $(p-1)\times(p-1)$ *matrix. Its ijth element is* $-(a_{i-j} + \beta_{i-j} + a_{i+j})$, *where* $a_k = 0$ *for* $k > p$ *or* $k < 0$, $a_0 = -1$ *and* $\beta_k = 0$ *for* $k \leqslant 0$ *or* $k > q$. *Moreover*, $\delta_i = q-1$ *for* $i \geqslant q$, *zero otherwise, and* $\gamma_i = 3$ *for* $i \geqslant q$, *one otherwise. Finally, the condition for the existence of the fourth moment is* $\delta_1 < 1$. □

3.1. The N-component GARCH(1,1) model

In what follows, we examine the N-component GARCH(1, 1) process, a special case of the GARCH model, introduced by DG (1996):

$$h_t = \sum_{i=1}^{n} w_i h_{it} \quad \text{where} \quad h_{it} = a_i \varepsilon_{t-1}^2 + \beta_i h_{it-1} + \gamma_i a_0, \tag{3.3}$$

where $\gamma_1 = 1$ and $\gamma_i = 0$ for $i \geqslant 2$.

Lemma 3.1. The above N-component GARCH(1, 1) *process can be expressed as a GARCH* (*n*, *n*) *process given by*

$$h_t = \sum_{i=1}^{n} w_i a_i \varepsilon_{t-1}^2 + \sum_{i=1}^{n} \sum_{l=2}^{n} \left[\prod_{j=1}^{l-1} \left(\sum_{i_j = i_j - 1 + 1, i_j \neq i}^{n-(l-1-j)} \right) \prod_{j=1}^{l-1} (\beta_{i_j})(-1)^{l+1} w_i a_i \varepsilon_{t-l}^2 \right]$$
$$+ \sum_{l=1}^{n} \left[\prod_{j=1}^{l} \left(\sum_{i_j = i_{j-1}+1}^{n-(l-j)} \right) \prod_{j=1}^{l} (\beta_{i_j})(-1)^{l+1} h_{t-l} \right] + w_1 a_0 \prod_{i=2}^{n} (1 - \beta_i). \tag{3.4}$$

Moreover, if in the N-component GARCH(1, 1) *process, Eq. (3.3), the first k* ($1 \leqslant k < n$) *components are integrated of order one* ($\beta_i = 1 - a_i$, $i = 1, \ldots, k$), *then the GARCH*(*n*, *n*) *process will still be stationary.* □

Once the N-component GARCH(1, 1) process is expressed as a GARCH(n, n) process, Theorems 2.1 and 3.1, can be used to obtain the second moment and the autocovariance function of the squared errors.

3.2. The two-component GARCH (n, n) model

We now consider another version of the GARCH model, the two-component GARCH(n, n) model:

$$h_t = w_1 h_{1t} + w_2 h_{2t}, \quad w_1 = 1 - w_2, \tag{3.5}$$

$$h_{1t} = a_0 + \sum_{i=1}^{n} a_{1i} \varepsilon_{t-i}^2 + \sum_{i=1}^{n} \beta_{1i} h_{1,t-i}, \quad h_{2t} = \sum_{i=1}^{n} a_{2i} \varepsilon_{t-i}^2 + \sum_{i=1}^{n} \beta_{2i} h_{2,t-i}. \tag{3.5a}$$

Lemma 3.2. The above two-component GARCH(n, n) model can be expressed as a GARCH (2n, 2n) process (without loss of generality, we ignore the constant), given by

$$h_t = \sum_{j=1}^{n} (w_1 a_{1j} + w_2 a_{2j})\, \varepsilon_{t-j}^2 - \sum_{j=1}^{n} \sum_{i=1}^{j} (w_1 a_{1,j+1-i} \beta_{2i} + w_2 a_{2,j+1-i} \beta_{1i})\, \varepsilon_{t-(j+1)}^2$$
$$- \sum_{j=2}^{n} \sum_{i=j}^{n} (w_1 a_{1,n+j-i} \beta_{2i} + w_2 a_{2,n+j-i} \beta_{1i}) \varepsilon_{t-(n+j)}^2 + \sum_{j=1}^{n} \sum_{i=1}^{j-1} [(\beta_{1j} + \beta_{2j})$$
$$- \beta_{1i} \beta_{2,j-i}] h_{t-j} - \sum_{j=1}^{n} \sum_{i=j}^{n} \beta_{1i} \beta_{2,n+j-i} h_{t-(n+j)}. \tag{3.6}$$

Moreover, if in the two-component GARCH(n, n) process, Eq. (3.5), *one component, h_{1t}* or *h_{2t}, is integrated of order one ($\sum_{j=1}^{n} \beta_{ij} = 1 - \sum_{j=1}^{n} a_{ij}$), then the GARCH(2n, 2n) process,* Eq. (3.6), *will still be stationary.* □

Having expressed the two-component GARCH(n, n) model as a GARCH($2n, 2n$) model, we can use Theorems 2.1 and 3.1 to obtain the second moment and the autocovariance function of the squared errors.

4. Concluding remarks

Since the observed volatility of an asset return is regarded as a realization of an underlying stochastic process it is not surprising that so much effort has been lavished on building models to measure and forecast it. The GARCH model and its various generalizations have been very popular in this respect and have been applied to various sorts of economic and financial data sets. However, there have been relatively fewer theoretical advancements. This paper has contributed to the theoretical developments in the GARCH literature. In Section 2 we presented a method for calculating the autocovariance function of the squared errors of the GARCH(p, q) model. Subsequently, in Section 3 we presented a method for calculating the fourth moment of the errors of the GARCH(p, q) model and we considered two extensions of the GARCH model. In particular, we expressed the N-component GARCH(1, 1) model as a GARCH(n, n) process and the two-component GARCH(n, n) model as a GARCH($2n, 2n$) process.

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Appendix A. Proof of Theorem 2.1

We prove the theorem by induction. If we assume that it holds for a GARCH($p-1$, q) process then it will be sufficient to prove that it holds for a GARCH(p, q) process.

If ε_t^2 is a GARCH(p, q) process given by Eq. (2.1) (for simplicity we will assume that $p-1 \geqslant q$) then it can be written as an ARCH(1) process with a GARCH($p-1$, q) error term (x_t) given by

$$\varepsilon_t^2 = a_1^* \varepsilon_{t-1}^2 + x_t \quad \text{where} \quad \prod_{i=2}^{p} (1 - a_i^* L) x_t = a_0 + v_t - \sum_{i=1}^{q} \beta_i v_{t-i}. \tag{A.1}$$

Note that the GARCH(p, q) process is symmetric with respect to the p roots $a_1^*, \ldots, a_p^*$. Hence, the autocovariance will also be symmetric with respect to the p roots.

Case (i): $0 \leqslant j \leqslant q-1$. Since x_t is a GARCH($p-1$, q) its autocovariance is given by

$$\begin{aligned} \mathrm{cov}_n(x_t) = \{ & \textstyle\sum_{i=2}^{p} \hat{e}_{in} z_{iq} \operatorname{var}(v_t) = \sum_{i=2}^{p} \hat{e}_{in}(z_{ij} - v_{ij}) \operatorname{var}(v_t), \quad \text{if } n \geqslant q, \\ & \textstyle\sum_{i=2}^{p} \hat{e}_{in} z_{i,j+b} \operatorname{var}(v_t) = \sum_{i=2}^{p} \hat{e}_{in}(z_{ij} + z_{ij,b^+}) \operatorname{var}(v_t), \text{ if} \\ & n = j+b, q-1-j \geqslant b \geqslant 1, \textstyle\sum_{i=2}^{p} \hat{e}_{in} z_{ij} \operatorname{var}(v_t), \text{ if} \\ & n = j, \textstyle\sum_{i=2}^{p} \hat{e}_{in} z_{i,j-b} \operatorname{var}(v_t) = \sum_{i=2}^{p} \hat{e}_{in}(z_{ij} + z_{ij,b^-}) \operatorname{var}(v_t), \text{ if} \\ & n = j-b, j \geqslant b \geqslant 1, \end{aligned} \tag{A.2}$$

where the $\hat{e}_{ij}$ and v_{ij} are given by

$$\hat{e}_{ij} = \frac{(a_i^*)^j (a_i^*)^{p-2}}{\prod_{l=2}^{p}(1 - a_i^* a_l^*) \prod_{k=2, k \neq i}^{p} (a_i^* - a_k^*)}, \tag{A.2a}$$

$$\begin{aligned} v_{ij} = & -\sum_{m=0}^{q-j-1} \beta_{q-m}[(a_i^*)^{q-2j-m} - (a_i^*)^{-q+m}] \\ & + \sum_{k=1}^{q-j-1} \sum_{l=0}^{q-j-(k+1)} \beta_k \beta_{q-l}[(a_i^*)^{q-2j-(l+k)} - (a_i^*)^{-q+l+k}]. \end{aligned} \tag{A.2b}$$

Note that, when $j = q-1$, the second term in the above equation vanishes since the lower limit exceeds the upper limit of the summation.

In addition, z_{ij} is given by Eq. (2.2b) and z_{ij,b^+} is given by

$$z_{ij,b^+} = \sum_{v=0}^{q-j-1}\left(-\beta_{q-v} + \sum_{l=1}^{v}\beta_l\beta_{q-(v-l)}\right)[(a_i^*)^{\pi_1(q-v)-2\pi_2(j+b)} - (a_i^*)^{q-2j-v}], \tag{A.3}$$

where $\pi_1 = 1$ for $v \leqslant q-j-b-1$, $\pi_1 = -1$ for $v \geqslant q-j-b$, $\pi_2 = 1$ for $v \leqslant q-j-b-1$, $\pi_2 = 0$ for $v \geqslant q-j-b$, $1 \leqslant j \leqslant q-2$, $1 \leqslant b \leqslant q-1-j$, and z_{ij,b^-} is given by

$$z_{ij,b^-} = \sum_{v=0}^{q+b-(j+1)}\left(-\beta_{q-v} + \sum_{l=1}^{v}\beta_l\beta_{q-(v-l)}\right) \times [(a_i^*)^{q+2b-(2j+v)} - (a_i^*)^{\pi_1(q-v)-2\pi_2 j}], \tag{A.4}$$

where $\pi_1 = 1$ for $0 \leqslant v \leqslant q-j-1$, $\pi_1 = -1$ for $v \geqslant q-j$, $\pi_2 = 1$ for $0 \leqslant v \leqslant q-j-1$, $\pi_2 = 0$ for $v \geqslant q-j$, and $2 \leqslant j \leqslant q-1$, $1 \leqslant b \leqslant j-1$.

Note that in Eqs. (A.3) and (A.4), when $v = 0$, the second summation term vanishes since the lower limit exceeds the upper limit of the summation operator.

In Eq. (A.2) we express the z_{iq}, $z_{i,j-b}$ and $z_{i,j+b}$ as functions of the z_{ij} coefficients.

The covariance between ε_t^2 and $x_{t-(j+v)}$ is given by

$$\varepsilon_t^2 = a_1^*\varepsilon_{t-1}^2 + x_t = \sum_{i=0}^{j+v-1}(a_1^*)^i x_{t-i} + \sum_{i=0}^{\infty}(a_1^*)^{j+v+i}x_{t-(j+v+i)} \Rightarrow \operatorname{cov}(\varepsilon_t^2, x_{t-(j+v)}) = \sum_{i=0}^{j+v-1}(a_1^*)^i\operatorname{cov}_{j+v-i}(x_t) + \sum_{i=0}^{\infty}(a_1^*)^{j+v+i}\operatorname{cov}_i(x_t). \tag{A.5}$$

Substituting Eq. (A.2) into the above equation, and after some algebra, we get

$$\operatorname{cov}(\varepsilon_t^2, x_{t-(j+v)}) = \left\{\sum_{i=2}^{p}\frac{\hat{e}_{i1}z_{ij}}{(a_i^* - a_1^*)}(a_i^*)^{j+v} + (a_1^*)^{j+v}\sum_{i=2}^{p}\hat{e}_{i0}z_{ij}\left(\frac{1}{1-a_i^*a_1^*} - \frac{a_i^*}{a_i^* - a_1^*}\right) - \sum_{i=2}^{p}\hat{e}_{iq}v_{ij}\left[\frac{(a_1^*)^{q+j+v}}{1-a_1^*a_i^*} + \pi_1\frac{(a_i^*)^{v+j-q+1} - (a_1^*)^{v+j-q+1}}{a_i^* - a_1^*}\right]\right.$$

$$+ (a_1^*)^v \sum_{k=v+1}^{q-1-j} \sum_{i=2}^{p} \hat{e}_{i,j+k} z_{ij,k^+} (a_1^*)^{2j+k}$$
$$+ (a_1^*)^v \sum_{k=1}^{j} \sum_{i=2}^{p} \hat{e}_{i,j-k} z_{ij,k^-} [\pi_2 (a_1^*)^k + (a_1^*)^{2j-k}]$$
$$+ (a_1^*)^v \sum_{k=1}^{\min(v,q-1-j)} \sum_{i=2}^{p} \hat{e}_{i,j+k} z_{ij,k^+} [(a_1^*)^{-k}$$
$$\left. + (a_1^*)^{2j+k}] \right\} \mathrm{var}(v_t), \tag{A.6}$$

where $\pi_1 = 1$ for $v > q-j-1$, $\pi_1 = 0$, otherwise, and $\pi_2 = 0$ for $k = j$, $\pi_2 = 1$, otherwise.

The jth autocovariance of ε_t^2 $(0 \leqslant j \leqslant q-1)$ is given by

$$\varepsilon_{t-j}^2 = \sum_{i=0}^{\infty} (a_1^*)^i x_{t-j-i} \Rightarrow \mathrm{cov}_j(\varepsilon_t^2) = \sum_{i=0}^{\infty} (a_1^*)^i \mathrm{cov}(\varepsilon_t^2, x_{t-j-i}). \tag{A.7}$$

Substituting Eq. (A.6) into the above equation, and after some algebra, we get

$$\mathrm{cov}_j(\varepsilon_t^2) = \left\{ \sum_{i=2}^{p} e_{ij} z_{ij} + (a_1^*)^j \zeta' \right\} \mathrm{var}(v_t), \tag{A.8}$$

where

$$\zeta' = \frac{1}{1-(a_1^*)^2} \sum_{k=1}^{j} \sum_{i=2}^{p} \hat{e}_{i,j-k} z_{ij,k^-} [\pi_1 (a_1^*)^{k-j} + (a_1^*)^{j-k}]$$
$$+ \sum_{v=0}^{q-1-j} \sum_{k=1}^{v} \sum_{i=2}^{p} (a_1^*)^{2v} \hat{e}_{i,j+k} z_{ij,k^+} [(a_1^*)^{-k-j} + (a_1^*)^{j+k}]$$
$$+ \frac{(a_1^*)^{2(q-j)}}{1-(a_1^*)^2} \sum_{k=1}^{q-1-j} \sum_{i=2}^{p} \hat{e}_{i,j+k} z_{ij,k^+} [(a_1^*)^{-k-j} + (a_1^*)^{j+k}]$$
$$+ \sum_{v=0}^{q-2-j} \sum_{k=v+1}^{q-1-j} \sum_{i=2}^{p} (a_1^*)^{2v} \hat{e}_{i,j+k} z_{ij,k^+} (a_1^*)^{j+k}$$
$$+ \sum_{i=2}^{p} \frac{\hat{e}_{i0} z_{ij}}{1-(a_1^*)^2} \left(\frac{1}{1-a_1^* a_i^*} - \frac{a_i^*}{a_i^* - a_1^*} \right)$$
$$- \sum_{i=2}^{p} \hat{e}_{iq} v_{ij} \left\{ \frac{(a_1^*)^q}{(1-a_1^* a_i^*)[1-(a_1^*)^2]} + \frac{(a_1^*)^{q-2j} a_i^*}{(a_i^* - a_1^*)(1-a_1^* a_i^*)} \right.$$
$$\left. - \frac{(a_1^*)^{q-2j+1}}{(a_i^* - a_1^*)[1-(a_1^*)^2]} \right\}, \tag{A.8a}$$

where $\pi_1 = 1$ for $k = j$, $\pi_1 = 0$ otherwise.

The autocovariance of the GARCH(p, q) process is equal to the sum of p terms. The last term is the product of $(a_1^*)^j$ and a coefficient which depends on $a_1^*, \ldots, a_p^*, \beta_1, \ldots, \beta_q$. Each of the first $p-1$ terms is the product of $(a_i^*)^j$, where $i = 2, \ldots, p$, and a coefficient which depends on $a_1^*, \ldots, a_p^*, \beta_1, \ldots, \beta_q$ $[e_{i0} \cdot z_{iq}]$. Given the symmetry of the autocovariance function in the inverse of the roots (i-roots) of the autoregressive polynomial, if we interchange the i-root a_i^* with the first i-root a_1 in the coefficient $e_{i0}z_{iq}$ we will get the coefficient of $(a_1^*)^j$, i.e. the coefficient ζ'. Hence, ζ' is given by

$$\zeta' = e_{10}z_{1j}. \tag{A.9}$$

Finally, substituting Eq. (A.9) into Eq. (A.8) we get Eq. (2.2). □

Case (ii): $j \geqslant q$. Since x_t is a GARCH($p-1$, q) process its autocovariance is given by

$$\operatorname{cov}_j(x_t) = \begin{cases} \sum_{i=2}^{p} \hat{e}_{ij} z_{ij} \operatorname{var}(v_t) = \sum_{i=2}^{p} \hat{e}_{ij} (z_{iq} + v_{ij}) \operatorname{var}(v_t), & 0 \leqslant j \leqslant q-1, \\ \sum_{i=2}^{p} \hat{e}_{ij} z_{iq} \operatorname{var}(v_t), & j \geqslant q, \end{cases} \tag{A.10}$$

.where $\hat{e}_{ij}$, v_{ij} are given by Eqs. (A.2a), (A.2b) and z_{iq} is given by Eq. (2.2b).

In Eq. (A.10) we expressed the z_{ij} coefficients as functions of the z_{iq} coefficients.

The covariance of x_t and ε_{t-j}^2 is given by

$$\operatorname{cov}(x_t, \varepsilon_{t-j}^2) = \sum_{i=0}^{\infty} (a_i^*)^i \operatorname{cov}(x_t, x_{t-(j+i)}). \tag{A.11}$$

Substituting Eq. (A.10) into the above equation we get

$$\operatorname{cov}(x_t, \varepsilon_{t-j}^2) = \left(\sum_{i=2}^{p} \frac{\hat{e}_{ij} z_{iq}}{1 - a_1^* a_i^*} + \pi \sum_{i=2}^{p} \hat{e}_{ij} f_{ij} \right) \operatorname{var}(v_t)$$
$$\text{where } f_{ij} = \sum_{v=j}^{q-1} v_{iv} (a_1^* a_i^*)^{v-j} \tag{A.12}$$

and $\pi = 0$ for $j \geqslant q$, $\pi = 1$ for $1 \leqslant j \leqslant q-1$.

The jth autocovariance of ε_t^2 is given by

$$\begin{aligned} \operatorname{cov}_j(\varepsilon_t^2) &= \sum_{l=0}^{j-1} \operatorname{cov}(x_{t-l}, \varepsilon_{t-j}^2)(a_1^*)^l + (a_1^*)^j \operatorname{var}(\varepsilon_t^2) \\ &= \sum_{v=1}^{j} \operatorname{cov}(x_t, \varepsilon_{t-v}^2)(a_1^*)^{j-v} + (a_1^*)^j \operatorname{var}(\varepsilon_t^2). \end{aligned} \tag{A.13}$$

Substituting Eq. (A.12) into the above equation, and after some algebra, we get

$$\operatorname{cov}_j(\varepsilon_t^2) = \left\{ \sum_{i=2}^{p} = e_{i0} z_{iq} (a_i^*)^j \left[1 - \left(\frac{a_1^*}{a_i^*} \right)^j \right] + \sum_{v=1}^{q-1} \sum_{i=2}^{p} (a_1^*)^{j-v} \hat{e}_{iv} f_{iv} \right\} \operatorname{var}(v_t) + (a_1^*)^j \operatorname{var}(\varepsilon_t^2)$$

$$= \sum_{i=2}^{p} e_{ij} z_{iq} \operatorname{var}(v_t) + (a_1^*)^j \zeta, \text{ where} \tag{A.14}$$

$$\zeta = \operatorname{var}(\varepsilon_t^2) + \left[\sum_{v=1}^{q-1} \sum_{i=2}^{p} (a_1^*)^{-v} \hat{e}_{iv} f_{iv} - \sum_{i=2}^{p} e_{i0} z_{iq} \right] \operatorname{var}(v_t). \tag{A.14a}$$

We employ the same reasoning with the one used for the $j \leqslant q-1$ case to get

$$\zeta = e_{10} z_{1q} \operatorname{var}(v_t). \tag{A.15}$$

Finally, substituting the above equation into Eq. (A.14) we get Eq. (2.2). □

Appendix B. Proof of Theorem 3.1

Using Eq. (3.1), the law of iterated expectations, and the following equations:

$$\lambda_i = E(h_t \varepsilon_{t-i}^2) = E(\varepsilon_t^2 \varepsilon_{t-i}^2), \quad c_i = E(h_t h_{t-i}) = E(\varepsilon_t^2 h_{t-i}),$$
$$E(\varepsilon_t^2 h_t) = E(h_t^2) = f_2/3 \tag{B.1}$$

we get

$$\frac{f_2}{3} = a_0 f_1 + \sum_{i=1}^{p} a_i \lambda_i + \sum_{i=1}^{q} \beta_i c_i, \tag{B.2}$$

where

$$c_j = a_0 f_1 + (a_j + \beta_j) \frac{f_2}{3} + \sum_{i=1}^{j-1} (\beta_{j-i} + a_{j-i}) c_i + \sum_{i=1}^{q-j} \beta_{j+i} c_i + \sum_{i=1}^{p-j} a_{j+i} \lambda_i, \tag{B.2a}$$

$$\lambda_j = a_0 f_1 + (3a_j + \beta_j)\frac{f_2}{3} + \sum_{i=1}^{j-1} (\beta_{j-i} + a_{j-i})\lambda_i$$

$$+ \sum_{i=1}^{q-j} \beta_{j+i} c_i + \sum_{i=1}^{p-j} a_{j+i}\lambda_i, \tag{B.2b}$$

$$\text{where } \beta_k = 0 \quad \text{for } k > q, \quad \text{and } a_k = 0 \quad \text{for } > k > p.$$

Note that in Eqs. (B.2a) and (B.2b) when $j = 1$, the first summation terms become zero, when $j \geqslant q$ the second summation terms become zero, and when $j \geqslant p$ the third summation terms become zero, since the lower limit exceeds the upper limit of the summations.

Substituting c_q and λ_p into Eq. (B.2) we get

$$\frac{f_2}{3} = a_0 f_1[1 + a_p + \beta_q] + \frac{f_2}{3}[3a_p^2 + \beta_q^2 + a_p\beta_p + a_q\beta_q]$$

$$+ \sum_{i=1}^{p-1} [a_i + a_p(a_{p-i} + \beta_{p-i})]\lambda_i + \sum_{i=1}^{q-1} [\beta_i + \beta_q(a_{q-i} + \beta_{q-i})]c_i$$

$$+ \beta_q \sum_{i=1}^{p-q} a_{q+i}\lambda_i + a_p \sum_{i=1}^{q-p} \beta_{p+i} c_i. \tag{B.3}$$

The system of Eqs. (B.2a) and (B.2b) can be written in matrix form as $A \cdot \bar{c} = B \cdot \bar{c}$ is a $(p + q - 2) \times 1$ column vector given by $\bar{c}' = [c_1 \cdots c_{q-1}\lambda_1 \cdots \lambda_{p-1}]$. B is a $(p + q - 2) \times 1$ column vector. Its $i1$th element is given by $a_0 f_1 + f_2/3(a_i + \beta_i)$ for $i \leqslant q - 1$, and its $q - 1 + i$, 1th element is given by $a_0 f_1 + (3a_i + \beta_i)f_2/3$ for $i \leqslant p - 1$.

Solving this system of equations we can express the c_i's and the λ_i's as functions of f_2.

$$c_j = a_0 f_1 \sum_{i=1}^{p+q-2} \Gamma_{ji} + \frac{f_2}{3} \sum_{i=1}^{p+q-2} (\gamma_i a_{i-\delta_i} + \beta_{i-\delta_i})\Gamma_{ji}, \tag{B.4a}$$

$$\lambda_j = a_0 f_1 = \sum_{i=1}^{p+q-2} \Gamma_{j+q-1,i} + \frac{f_2}{3} \sum_{i=1}^{p+q-2} (\gamma_i a_{i-\delta_i} + \beta_{i-\delta_i}) - \Gamma_{j+q-1,i}, \tag{B.4b}$$

where $\delta_i = q - 1$ for $i \geqslant q$, zero otherwise, and $\gamma_i = 3$ for $i \geqslant q$, one otherwise. Substituting Eqs. (B.4a) and (B.4b) into Eq. (B.3) and solving for f_2 we get $f_2 = 3a_0 f_1 \phi_1/(1 - \delta_1)$. □

Appendix C. Proof of Lemmas 3.1 and 3.2

Substitution of the h_{it}'s into the h_t of Eq. (3.3) yields

$$h_t = \sum_{i=1}^{n} w_i a_i \varepsilon_{t-1}^2 + \sum_{i=1}^{n} w_i \beta_i h_{i,t-1} + w_1 a_0. \tag{C.1}$$

In the above equation, when we add and subtract sequentially the following terms:

$$\sum_{i=1}^{n} \prod_{j=1}^{k} \left(\sum_{i_j = i_{j-1}+1,\ i_j \neq i}^{n-(k-j)} \right) \prod_{j=1}^{k} (\beta_{i_j}) w_i h_{i,\,t-k}, \quad 1 \leqslant k \leqslant n-1, \quad i_0 = 0, \tag{C.2}$$

we get Eq. (3.4). □

The sum of the coefficients of the ε_{t-i}^2's terms and of the h_{t-i}'s terms ($i = 1,2, \dots, n$) are

$$1 - \sum_{i=k+1}^{n} \prod_{j=k+1, j\neq i}^{n} (1-\beta_j) \prod_{l=1}^{k} (a_l) w_i (1 - a_i - \beta_i) < 1 \tag{C.3}$$

We substitute Eq. (3.5a) into Eq. (3.5) and we get (for simplicity we will assume that $a_0 = 0$)

$$h_t = \sum_{i=1}^{n} (w_1 a_{1i} + w_2 a_{2i}) \varepsilon_{t-i}^2 + w_1 \sum_{i=1}^{n} \beta_{1i} h_{1,t-i} + w_2 \sum_{i=1}^{n} \beta_{2i} h_{2,t-i}. \tag{C.4}$$

In the above equation, when we add and subtract sequencially the following terms:

$$w_1 \beta_{2k} h_{1,t-k} + w_2 \beta_{1k} h_{2,t-k}, \quad 1 \leqslant k \leqslant n, \tag{C.5}$$

we get Eq. (3.6). □

The sum of the coefficients of the ε_{t-i}^2's and of the h_{t-i}'s terms ($i = 1, 2, \dots, 2n$) are

$$w_1 \left[\left(1 - \sum_{i=1}^{n} \beta_{2i}\right) \left(\sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} \beta_{1j} \right) + \sum_{i=1}^{n} \beta_{2i} \right] + w_2 \left[\left(1 - \sum_{i=1}^{n} \beta_{1i}\right) \left(\sum_{j=1}^{n} a_{2j} + \sum_{j=1}^{n} \beta_{2j} \right) + \sum_{i=1}^{n} \beta_{1i} \right] < 1 \tag{C.6}$$

If $\sum_{i=1}^{n} a_{1i} + \sum_{i=1}^{n} \beta_{1i} = 1$ (i.e. the first component is integrated of order one), then the sum of the coefficients of ε_{t-i}^2 and h_{t-i} is

$$1 - w_2 \sum_{i=1}^{n} a_{1i}\left(1 - \sum_{j=1}^{n} a_{2j} - \sum_{j=1}^{n} \beta_{2j}\right) < 1 \quad \Box \tag{C.7}$$

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