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ARCH-type bilinear models with double long memory

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Abstract

We discuss the covariance structure and long-memory properties of stationary solutions of the bilinear equation $X_t = \zeta_t A_t + B_t, (\star)$, where $\zeta_t,\ t \in \mathbb{Z}$ are standard i.i.d. r.v.'s, and A_t, B_t are moving averages in $X_s,\ s < t$. Stationary solution of $(\star)$ is obtained as an orthogonal Volterra expansion. In the case $A_t \equiv 1$, X_t is the classical AR(∞) process, while $B_t \equiv 0$ gives the LARCH model studied by Giraitis et al. (Ann. Appl. Probab. 10 (2000) 1002). In the general case, X_t may exhibit long memory both in conditional mean and in conditional variance, with arbitrary fractional parameters $0 < d_1 < \frac{1}{2}$ and $0 < d_2 < \frac{1}{2}$, respectively. We also discuss the hyperbolic decay of auto- and/or cross-covariances of X_t and X_t^2 and the asymptotic distribution of the corresponding partial sums' processes.

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1. Introduction and the main results

The present paper studies the covariance structure and long-memory properties of a class of discrete time stationary processes which satisfy the bilinear equation

$$X_t = \zeta_t \left(a + \sum_{j=1}^{\infty} a_j X_{t-j} \right) + b + \sum_{j=1}^{\infty} b_j X_{t-j}, \tag{1.1}$$

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where $\{\zeta_t,\ t\in\mathbb{Z}\}$ are i.i.d. random variables ("shocks") with zero mean and variance 1, and $a,b,a_j,b_j,\ j\geqslant 1$ are real coefficients satisfying some conditions (which imply in particular that the series in (1.1) converge in mean square and X_t (1.1) is measurable w.r.t. to the σ-field generated by $\zeta_s,\ s\leqslant t$, see Section 2). Put

$$A_t = a + \sum_{j=1}^{\infty} a_j X_{t-j}, \quad B_t = b + \sum_{j=1}^{\infty} b_j X_{t-j}. \tag{1.2}$$

Then B_t and A_t^2 is the conditional mean and the conditional variance of X_t, respectively, i.e.

$$B_t = E\{X_t | X_s,\ s<t\}, \quad A_t^2 = \mathrm{var}\{X_t|X_s,\ s<t\}. \tag{1.3}$$

Recall that a time series is called *conditionally homoskedastic* or *conditionally heteroskedastic* depending on whether its conditional variance is constant or not. As a particular case, Eq. (1.1) includes the classical AR(∞) processes, which are defined as stationary solutions of (1.1) with $a_j\equiv 0$:

$$X_t - b - \sum_{j=1}^{\infty} b_j X_{t-j} = a\zeta_t. \tag{1.4}$$

In the case $b=b_j\equiv 0$, (1.1) is the *Linear ARCH* (*LARCH*) model introduced by Robinson (1991) and recently studied in Giraitis et al. (2000b), Giraitis et al. (2001):

$$X_t = \zeta_t \left(a + \sum_{j=1}^{\infty} a_j X_{t-j} \right). \tag{1.5}$$

Another particular case of (1.1) is the ARCH(∞) model:

$$r_t^2 = \varepsilon_t^2 \sigma_t^2, \quad \sigma_t^2 = c + \sum_{j=1}^{\infty} c_j r_{t-j}^2, \tag{1.6}$$

where $c_j,\ c_j\geqslant 0$ are non-negative parameters and $\{\varepsilon_t,\ t\in\mathbb{Z}\}$ are i.i.d. r.v.'s with zero mean and finite variance (Robinson, 1991; Giraitis et al., 2000a), which can be rewritten in the form (1.1) with $X_t=r_t^2$ and standardized zero mean $\zeta_t=(\varepsilon_t^2-E\varepsilon_t^2)/(\mathrm{var}(\varepsilon_0^2))^{1/2}$ (see Section 3 below). Eq. (1.6) includes the classical ARCH(p) and GARCH(p,q) models of Engle (1982) and Bollerslev and Mikkelsen (1996).

The general bilinear model (1.1) combines the dependence structure and properties of both linear (AR) and nonlinear (ARCH) models. In particular, (1.1) may exhibit *long memory* both *in conditional mean* and *in conditional variance*, with arbitrary (fractional) parameters $0<d_1,d_2<\frac{1}{2}$. As shown in the present paper, the parameters d_1,d_2 determine the decay rate of autocovariances of $\{X_t\}$ and $\{X_t^2\}$ and the behavior of the corresponding partial sums' processes.

The interest in models of heteroskedastic time series with long memory exists in econometrics and finance, where empirical facts about asset returns and some other financial data motivated the study of stationary processes which exhibit long memory in conditional variance. A number of such models (FIGARCH, LM-ARCH, FIEGARCH) were proposed in the ARCH literature; however, long-memory properties of some of these models have not been theoretically established, and even the existence of stationary solution remains controversial (Giraitis et al., 2000a; Mikosch and Stărică, 1999;

Kazakevičius et al., 2001). The long memory, in the sense of the asymptotic behavior of the covariance function, was rigorously established for a class of stochastic volatility models which includes Gaussian subordinated models with arbitrary form of nonlinearity (Robinson, 1999) and a general class of exponential volatily models and the EGARCH model (Surgailis and Viano, 2001).

As far as ARCH models have zero conditional mean, attempts have been made to generalize them to include non-zero drift (Baillie et al., 1996; Ling and Li, 1997; Teyssière, 2000). The last paper introduces a class of double long memory models which combine long memory ARCH and linear ARFIMA processes and discusses various inference procedures and Monte-Carlo simulations.

Let us describe the main results of the paper and the contents of the remaining sections. Section 2 obtains the stationary solution of (1.1) as orthogonal Volterra series. Let $A(z) := \sum_{j=1}^{\infty} a_j z^j$, $B(z) := \sum_{j=1}^{\infty} b_j z^j$, $|z| < 1$ be the generating functions of $\{a_j\}$ and $\{b_j\}$, respectively. We assume that $A(z)$ and $B(z)$ are analytic on $\{|z| < 1\}$ and $B(z) \neq 1$ ($|z| < 1$). Put

$$G(z) := (1 - B(z))^{-1} = \sum_{j=0}^{\infty} g_j z^j, \quad H(z) := A(z)(1 - B(z))^{-1} = \sum_{j=1}^{\infty} h_j z^j. \tag{1.7}$$

Put $\ell^p = \{\phi = (\phi_0, \phi_1, \ldots): \|\phi\|_p < \infty\}, \|\phi\|_p = \{\sum_{j=0}^{\infty} |\phi_j|^p\}^{1/p}$, $\|\phi\| := \|\phi\|_2$. Let $(\phi \star \psi)_j = \sum_{i=0}^{j} \phi_i \psi_{j-i}$ denote the convolution.

Assumption A_1. $\{g_j\} \in \ell^2$, $\{h_j\} \in \ell^2$ and

$$\|h\| = \left\{ \sum_{j=1}^{\infty} h_j^2 \right\}^{1/2} < 1. \tag{1.8}$$

Assumption A_2.

$$\|(a^{(n)} - a) \star g\| \to 0, \quad \|(b^{(n)} - b) \star g\| \to 0, \tag{1.9}$$

where $a_j^{(n)} := a_j I\ (1 \leqslant j \leqslant n)$, $b_j^{(n)} := b_j I\ (1 \leqslant j \leqslant n)$.

According to Theorem 2.2 below, if Assumptions A_1 and A_2 are satisfied, then Eq. (1.1) with $b=0$ admits the stationary solution $X_t = \zeta_t A_t + B_t$, with $A_t = a + A_t^0$, $B_t = B_t^0$ and $A_t^0 := \sum_{j=1}^{\infty} a_j X_{t-j}$, $B_t^0 := \sum_{j=1}^{\infty} b_j X_{t-j}$ given by the convergent orthogonal Volterra series

$$A_t^0 = a \sum_{k=1}^{\infty} \sum_{s_k < \cdots < s_1 < t} h_{t-s_1} h_{s_1 - s_2} \ldots h_{s_{k-1} - s_k} \zeta_{s_1} \ldots \zeta_{s_k}, \tag{1.10}$$

$$B_t^0 = a \sum_{k=1}^{\infty} \sum_{s_k < \cdots < s_1 < t} g_{t-s_1} h_{s_1 - s_2} \ldots h_{s_{k-1} - s_k} \zeta_{s_1} \ldots \zeta_{s_k}. \tag{1.11}$$

The above solution can be written in the more compact form:

$$X_t = a \sum_{k=1}^{\infty} \sum_{s_k < \cdots < s_1 \leqslant t} g_{t-s_1} h_{s_1 - s_2} \ldots h_{s_{k-1} - s_k} \zeta_{s_1} \ldots \zeta_{s_k}. \tag{1.12}$$

In the case $b=b_j=0$, one has $g_j=\delta_{0j}$, $h_j=a_j$, where $\delta_{0j}=1$ if $j=0$, $=0$ otherwise, and (1.12) coincides with the stationary solution of the LARCH equation (1.5) given in Giraitis et al. (2000b).

Theorem 2.4 obtains a similar representation of a stationary solution of (1.1) with $b\neq 0$.

Assumption A$_3$. There exist finite limits $\bar{a}:=\lim_{n\to\infty}\sum_{j=1}^{n} a_j$ and $\bar{b}:=\lim_{n\to\infty}\sum_{j=1}^{n} b_j$ such that

$$\bar{b}\neq 1. \tag{1.13}$$

Under Assumptions A$_1$–A$_3$, Eq. (1.1) admits the stationary solution

$$X_t=\mu+(\bar{a}+\mu\bar{a})\sum_{k=1}^{\infty}\sum_{s_k<\cdots<s_1\leqslant t} g_{t-s_1}h_{s_1-s_2}\dots h_{s_{k-1}-s_k}\zeta_{s_1}\dots\zeta_{s_k}, \tag{1.14}$$

where $\mu=EX_t=b/(1-\bar{b})$ (Theorem 2.4). Eqs. (1.12) or (1.14) imply the useful formula for the covariance

$$\operatorname{cov}(X_0,X_t)=\frac{(a+\mu\bar{a})^2}{1-\|h\|^2}\sum_{j=0}^{\infty} g_j g_{j+t}. \tag{1.15}$$

Eq. (1.14) also yields an *orthogonal* Volterra series representation for the stationary solution $X_t=r_t^2$ of the ARCH(∞) Eq. (1.6) and a new sufficient and necessary condition for the existence of its covariance stationary solution (see Section 3).

Section 4 discusses long-memory properties of the stationary solution of (1.1) with $b=0$. Eqs. (1.10)–(1.12) imply the moving average representations

$$A_t=a+\sum_{j=1}^{\infty} h_j Y_{t-j},\quad B_t=\sum_{j=1}^{\infty} g_j Y_{t-j},\quad X_t=\sum_{j=0}^{\infty} g_j Y_{t-j} \tag{1.16}$$

with respect to the weak white noise (martingale difference sequence) $\{Y_s:=\zeta_s A_s\}$. As it turns out, the stationary solution $X_t=\zeta_t A_t+B_t$ may exhibit long memory both in conditional mean and in conditional variance, in the sense that the transfer functions $G(z)$ and $H(z)$ admit the representations

$$G(z)=P_1(z)(1-z)^{-d_1},\quad H(z)=P_2(z)(1-z)^{-d_2}, \tag{1.17}$$

where $0<d_i<\frac{1}{2}$, $i=1,2$, and $P_i(z)=\sum_{j=0}^{\infty} p_{ij}z^j$, $i=1,2$ have no poles on the unit disk $|z|\leqslant 1$. Under additional conditions on $P_i(z)$, $i=1,2$, (1.17) implies that the autocovariance functions of $\{A_t\}$ and $\{B_t\}$ decay as t^{2d_2-1} and t^{2d_1-1}, respectively. In Section 4 we also present concrete examples of $\{a_j\}$ and $\{b_j\}$ satisfying (1.17) together with Assumptions A$_1$ and A$_2$.

Section 5 discusses the decay of auto- and/or cross-covariance functions of the "observable" sequences $\{X_t\}$ and $\{X_t^2\}$. Assuming (1.17) and some additional moment conditions, we show that $\operatorname{cov}(X_0,X_t)$ and $\operatorname{cov}(X_0,X_t^2)$ decay as t^{2d_1-1} and $t^{d_1+d_2-1}$, respectively. For the LARCH model, the last decay was obtained in Giraitis et al. (2001).

The asymptotic behavior of $\mathrm{cov}(X_0^2, X_t^2)$ is more complicated:

$$\mathrm{cov}(X_0^2, X_t^2) \sim \begin{cases} \kappa'_{22}\, t^{2(2d_1-1)} & \text{if } 2(1-2d_1) < 1-2d_2, \\ \kappa''_{22}\, t^{2d_2-1} & \text{if } 2(1-2d_1) > 1-2d_2, \end{cases} \tag{1.18}$$

where $\kappa'_{22}, \kappa''_{22}$ are some constants (Theorem 5.1). The dichotomy in the asymptotic behavior (1.18) is reflected in the asymptotic distribution of suitably normalized sums $\sum_{t=1}^{N}(X_s^2 - EX_s^2)$, which is Gaussian for $2(1-2d_1) > 1-2d_2$, and non-Gaussian (given by a double Ito–Wiener integral) for $2(1-2d_1) < 1-2d_2$ $(0 < d_1, d_2 < \frac{1}{2})$ (Theorem 6.1). On the other hand, linear sums $\sum_{t=1}^{N} X_s$ are asymptotically Gaussian under normalization which depends only on d_1 (Theorem 6.2).

Let us note, finally, that formally (1.1) appears to be a particular case of more general bilinear models introduced by Granger and Andersen (1978) and later studied by several authors (see Tong, 1981; Subba Rao and Gabr, 1984; Terdik, 1999). However, these works seem to focus on bilinear models with short memory which specifically exclude (1.1). Continuous time bilinear analogs of (1.1) also have been studied in the literature, see e.g. Ito and Nisio (1964), Morozan (1996).

2. Existence of stationary solution

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $\{\zeta_s,\ s \in \mathbb{Z}\}$ be a sequence of i.i.d. r.v.'s defined on this space, with zero mean and unit variance. Let $\mathscr{F}_t = \sigma\{\zeta_s,\ s \leqslant t\}$, $t \in \mathbb{Z}$ be the increasing family of sub-σ-fields of $\mathscr{F}$. A random sequence $\{y_t,\ t \in \mathbb{Z}\}$ is called *adapted* if, for each $t \in \mathbb{Z}$, y_t is $\mathscr{F}_t$-measurable. Let $L^p(\Omega)$ $(1 \leqslant p < \infty)$ denote the Banach space of all complex-valued random variables ξ defined on $(\Omega, \mathscr{F}, P)$ such that $E|\xi|^p < \infty$ (we identify r.v.'s which coincide P-a.s.). Write l.i.m. for the limit in mean square.

Definition 2.1. By a solution of (1.1) we mean an adapted sequence $\{X_t,\ t \in \mathbb{Z}\}$ with finite second moment $EX_t^2 < \infty$, such that for every $t \in \mathbb{Z}$, the series $A_t^0 = \sum_{j=1}^{\infty} a_j X_{t-j}$ and $B_t^0 = \sum_{j=1}^{\infty} b_j X_{t-j}$ converge in mean square and (1.1) holds.

Note the above definition implies the convergence $X_t = \text{l.i.m.}\,[\zeta_t(a + \sum_{j=1}^{n} a_j X_{t-j}) + b + \sum_{j=1}^{n} b_j X_{t-j}]$ in $L^2(\Omega)$.

Theorem 2.2. *Let Assumptions* A_1 *and* A_2 *be satisfied and let* $b=0$. *Then there exists a solution of* (1.1) *which is unique, strictly stationary, ergodic and is given by the convergent orthogonal Volterra series* (1.12). *Moreover,* $EX_t = 0$ *and*

$$\mathrm{cov}(X_0, X_t) = \frac{a^2}{1-\|h\|^2} \sum_{j=0}^{\infty} g_j g_{j+t}. \tag{2.1}$$

Proof. Let us check that series (1.12) converges. By orthogonality,

$$EX_t^2 = a^2 \sum_{k=1}^{\infty} \sum_{s_k < \cdots < s_1 \leqslant t} g_{t-s_1}^2 h_{s_1-s_k}^2 \ldots h_{s_{k-1}-s_k}^2$$

$$= a^2 \|g\|^2 \sum_{k=0}^{\infty} \|h\|^{2k} = a^2 \|g\|^2/(1 - \|h\|^2). \tag{2.2}$$

Eq. (2.1) follows similarly. Clearly, $\{X_t\}$ of (1.12) is strictly stationary and adapted.

Let us show that (1.12) is a solution of (1.1). Put

$$A_{t,n}^0 := \sum_{j=1}^{n} a_j X_{t-j}, \quad B_{t,n}^0 := \sum_{j=1}^{n} b_j X_{t-j}.$$

Let us show that $A_{t,n}^0$ and $B_{t,n}^0$ converge in $L^2(\Omega)$ to random variables A_t^0 and B_t^0 defined in (1.10) and (1.11), respectively. By (1.12),

$$A_t^0 - A_{t,n}^0 = a \sum_{k=1}^{\infty} \sum_{s_k < \cdots < s_1 < t} \left\{ h_{t-s_1} - \sum_{j=1}^{n \wedge (t-s_1)} a_j g_{t-j-s_1} \right\} h_{s_1-s_2} \ldots h_{s_{k-1}-s_k} \zeta_{s_1} \ldots \zeta_{s_k}.$$

The expression in $\{\ldots\}$ equals $h_{t-s_1} - (a^{(n)} \star g)_{t-s_1} = ((a - a^{(n)}) \star g)_{t-s_1}$ and we obtain as in (2.2)

$$E(A_t^0 - A_{t,n}^0)^2 = a^2 \|(a - a^{(n)}) \star g\|^2/(1 - \|h\|^2).$$

Thus the convergence of $A_{t,n}^0$ to A_t^0 follows by Assumption A_2. Similarly,

$$B_t^0 - B_{t,n}^0 = a \sum_{k=1}^{\infty} \sum_{s_k < \cdots < s_1 < t} \left\{ g_{t-s_1} - \sum_{j=1}^{n \wedge (t-s_1)} b_j g_{t-j-s_1} \right\} h_{s_1-s_2} \ldots h_{s_{k-1}-s_k} \zeta_{s_1} \ldots \zeta_{s_k}.$$

As $G(z) = 1 + B(z)G(z)$, or $g_t = (b \star g)_t$ $(t \geqslant 1)$, the expression inside the curly brackets equals $g_{t-s_1} - (b^{(n)} \star g)_{t-s_1} = ((b - b^{(n)}) \star g)_{t-s_1}$, implying

$$E(B_t^0 - B_{t,n}^0)^2 = a^2 \|(b - b^{(n)}) \star g\|^2/(1 - \|h\|^2)$$

and the convergence of $B_{t,n}^0$ follows again by Assumption A_2. As $X_t = \zeta_t(a + A_t^0) + B_t^0$, see (1.10)–(1.12), this proves that $\{X_t\}$ of (1.12) is a solution of (1.1). The ergodicity follows from Stout (1974, Theorem 3.5.8), as $X_t = f(\zeta_t, \zeta_{t-1}, \ldots)$ for a measurable f.

It remains to show the uniqueness. Let X_t', X_t'' be two solutions of (1.1). Then $\tilde{X}_t := X_t' - X_t''$ is a solution of

$$\tilde{X}_t = \zeta_t \sum_{j=1}^{\infty} a_j \tilde{X}_{t-j} + \sum_{j=1}^{\infty} b_j \tilde{X}_{t-j} =: \zeta_t \tilde{A}_t^0 + \tilde{B}_t^0,$$

where the series $\tilde{A}_t^0, \tilde{B}_t^0$ converge in L^2. As $\tilde{X}_t - \sum_{j=1}^{\infty} b_j \tilde{X}_{t-j} = \tilde{Y}_t$, where $\tilde{Y}_t := \zeta_t \tilde{A}_t^0$ are uncorrelated with zero mean and variance $E\tilde{Y}_t^2 = E(\tilde{A}_t^0)^2 < \infty$, by inverting this representation one has

$$\tilde{X}_t = (1 - B(L))^{-1} \tilde{Y}_t = \sum_{j=0}^{\infty} g_j \tilde{Y}_{t-j}$$

and therefore, using $h_j = (a \star g)_j$,

$$\tilde{A}_t^0 = \sum_{j=1}^{\infty} (a \star g)_j \tilde{Y}_{t-j} = \sum_{j=1}^{\infty} h_j \tilde{Y}_{t-j}.$$

Hence

$$E(\tilde{A}_t^0)^2 = \|h\|^2 E\tilde{Y}_t^2 = \|h\|^2 E(\tilde{A}_t^0)^2$$

or $\|h\| = 1$. But this contradicts (1.8), hence $E(\tilde{A}_t^0)^2 = 0$ and $E\tilde{X}_t^2 = \|g\|^2 E(\tilde{A}_t^0)^2 = 0$. □

Remark 2.3. From Theorem 3.2 it follows that the homogeneous Eq. (1.1) with $a{=}b{=}0$ under Assumptions A_1 and A_2 admits only trivial solution $X_t \equiv 0$.

Next, we discuss the case $b \neq 0$.

Theorem 2.4. *Let* $b \neq 0$, *and let Assumptions* A_1–A_3 *be satisfied. Then Eq.* (1.1) *admits a solution which is unique, strictly stationary, ergodic and is given by the convergent Volterra series* (1.14). *Moreover,* $EX_t = \mu$ *and the covariance of* $\{X_t\}$ *is given by* (1.15).

Proof. Let $\bar{X}_t$ be a solution to the equation

$$\bar{X}_t = \zeta_t \left(a + \bar{a}\mu + \sum_{j=1}^{\infty} a_j \bar{X}_{t-j} \right) + \sum_{j=1}^{\infty} b_j \bar{X}_{t-j}. \tag{2.3}$$

According to Theorem 2.2, such a solution exists, is unique, and is written as the Volterra series (1.12) with a replaced by $a + \bar{a}\mu$. Then $X_t := \mu + \bar{X}_t$ is a solution to (1.1). Indeed,

$$\mu + \bar{X}_t = \mu + \text{l.i.m.}\, \zeta_t \left(a + \bar{a}\mu + \sum_{j=1}^{n} a_j \bar{X}_{t-j} \right) + \text{l.i.m.} \sum_{j=1}^{n} b_j \bar{X}_{t-j}$$

$$= \text{l.i.m.}\, \zeta_t \left(a + \sum_{j=1}^{n} a_j (\mu + \bar{X}_{t-j}) \right) + \text{l.i.m.} \sum_{j=1}^{n} b_j (\mu + \bar{X}_{t-j}).$$

The remaining statements of Theorem 2.4 including representation (1.14) follow now from Theorem 2.2. □

Remark 2.5. From Theorem 2.4 it follows that under Assumptions A_1–A_3 Eq. (1.1) with $b \neq 0$ and $a + \bar{a}\mu = 0$ admits the unique trivial solution $X_t \equiv \mu$.

Remark 2.6. Assumption A_3 is necessary for the existence of a solution of (1.1) with constant mean $\mu{=}EX_t \neq 0$. This fact follows from Definition 3.1, the existence of finite limits $\lim E(\sum_{j=1}^{n} a_j X_{t-j}) = \mu \lim \sum_{j=1}^{n} a_j = \mu\bar{a}$, $\lim E(\sum_{j=1}^{n} b_j X_{t-j}) = \mu \lim \sum_{j=1}^{n} b_j = \mu\bar{b}$ and the identity $EX_t = b + \bar{b}EX_t$.

Proposition 2.7. *Assume* $\{a_j\} \in \ell^2$, *the existence of finite limit* $\lim \sum_{j=1}^n a_j = \bar{a}$ *if* $b \neq 0$, *and either* $\sum_{j=1} |b_j| < 1$, *or* $\sum_{j=1}^\infty |b_j| < \infty$ *and* $|B(z)| < 1$ *for all* $|z| \leqslant 1$. *Then*:

(i) *Assumptions* A_2 *and* A_3 *are satisfied, as well as conditions* $\{g_j\} \in \ell^2$, $\{h_j\} \in \ell^2$ *of Assumption* A_1 *with exception of* $\|h\| < 1$.

(ii) *If, in addition,* $\|h\| < 1$ *holds, then the statements of Theorems* 2.2 *and* 2.4 *apply and the solution* X_t *of* (1.1) *has absolutely summable covariances*

$$\sum_{t \in \mathbb{Z}} |\mathrm{cov}(X_0, X_t)| < \infty. \tag{2.4}$$

(iii) $\|a\| + \|b\|_1 < 1$ *implies* $\|h\| < 1$.

Proof. (i) The assumptions on $\{b_j\}$ imply $\{g_j\} \in \ell^1$; see Rudin (1987) and Giraitis et al. (2000a, Lemma 4.1). Whence and from the conditions of the proposition it follows all requirements of Assumptions A_1–A_3 with exception of $\|h\| < 1$.

(ii) Eq. (2.4) follows from $\{g_j\} \in \ell^1$ and (1.15), (2.1).

(iii) Follows from $\|h\| \leqslant \|a\| \|g\|_1$ and $\|g\|_1 \leqslant (2\pi)^{-1} \int_{-\pi}^{\pi} |1 - B(\mathrm{e}^{-\mathrm{i}\lambda})|^{-1} \, \mathrm{d}\lambda \leqslant 1/(1 - \|b\|_1)$. □

Example 2.8. Consider the equation

$$X_t = \zeta_t(1 + \alpha X_{t-1}) + \beta X_{t-1}, \tag{2.5}$$

where α, β are real parameters. In this case, $g_t = \beta^t$ $(t \geqslant 0)$ and $h_t = \alpha\beta^{t-1}$ $(t \geqslant 1)$. Condition $\|h\| < 1$ in this case becomes $\|h\|^2 = \alpha^2/(1 - \beta^2) < 1$, or

$$\alpha^2 + \beta^2 < 1. \tag{2.6}$$

Clearly, (2.6) implies Assumptions A_1 and A_2. According to Theorem 2.2, the stationary solution X_t of (2.5) is

$$X_t = \sum_{k=1}^{\infty} (\alpha/\beta)^k \sum_{s_k < \cdots < s_1 \leqslant t} \beta^{t - s_k} \zeta_{s_1} \cdots \zeta_{s_k}.$$

Another form of the solution can be obtained by direct iteration of (2.5):

$$X_t = \zeta_t + \sum_{j=1}^{\infty} \zeta_{t-j} \prod_{s=t-j+1}^{t} (\alpha\zeta_s + \beta). \tag{2.7}$$

Note that the series in (2.7) is *orthogonal* and convergent in $L^2(\Omega)$:

$$EX_t^2 = 1 + \sum_{j=1}^{\infty} E\left(\prod_{s=t-j+1}^{t} (\alpha\zeta_s + \beta)\right)^2 = 1 + \sum_{j=1}^{\infty} (\alpha^2 + \beta^2)^j = (1 - \alpha^2 - \beta^2)^{-1}.$$

A continuous time version of (2.5) is the stochastic differential equation

$$\mathrm{d}X_t = (1 + \alpha X_t) \, \mathrm{d}W_t + \beta X_t \, \mathrm{d}t, \quad t \in \mathbb{R},$$

where W_t, $t \in \mathbb{R}$ is standard Brownian motion. The last equation can be explicitly solved: $X_t = \int_{-\infty}^t \exp\{\alpha(W_t - W_s) + (\beta - \alpha^2/2)(t - s)\} \, \mathrm{d}W_s$, for $\beta < -\alpha^2/2$. See Ito and Nisio (1964), Morozan (1996) on stationary solutions of more general bilinear equations with continuous time.

3. Covariance stationary ARCH(∞) sequences

According to Giraitis et al. (2000a), a random sequence $\{X_t,\ t\in\mathbb{Z}\}$ satisfies an ARCH(∞) equation if there exist a sequence $\{\varepsilon_t,\ t\in\mathbb{Z}\}$ of i.i.d. random variables and (non-random) numbers $c,\ c_j\geqslant 0\ (j\geqslant 1)$ such that

$$X_t=\varepsilon_t^2\left(c+\sum_{j=1}^{\infty}c_jX_{t-j}\right). \tag{3.1}$$

The problem of the existence of a stationary solution of Eq. (3.1), with possibly infinite mean EX_0, was studied by Nelson (1990), Bougerol and Picard (1992) and recently by Kazakevičius et al. (2001).

Conditions for the existence of covariance stationary solutions of (3.1) were obtained in Embrechts et al. (1997), Giraitis et al. (2000a), Kazakevičius et al. (2001). Put $\lambda_i=E\varepsilon_0^{2i},\ i=1,2;\ \sigma^2=\mathrm{var}(\varepsilon_0^2)=\lambda_2-\lambda_1^2$. As shown in Giraitis et al. (2000a) (see also Kazakevičius et al., 2001), condition

$$\lambda_1\sum_{j=1}^{\infty}c_j<1 \tag{3.2}$$

is necessary and sufficient for the existence of a strictly stationary solution of (3.1) with finite expectation $EX_t<\infty$. Moreover, the solution is unique and can be written as

$$X_t=c\varepsilon_t^2\sum_{k=0}^{\infty}\sum_{s_k<\cdots<s_1<t}c_{t-s_1}\ldots c_{s_{k-1}-s_k}\varepsilon_{s_1}^2\ldots\varepsilon_{s_k}^2, \tag{3.3}$$

the series convergent in $L^1(\Omega)$, with mean $\mu:=EX_t=c\lambda_1/(1-\lambda_1\sum_{j=1}^{\infty}c_j)$. In the degenerate case $\sigma=0$, or $\varepsilon_t^2=\lambda_1$ a.s., (3.3) becomes $X_t=\mu$ a.s. Giraitis et al. (2000a) also showed that

$$\lambda_2^{1/2}\sum_{j=1}^{\infty}c_j<1 \tag{3.4}$$

is sufficient in order that the solution X_t (3.3) has finite variance (and therefore is covariance stationary).

By introducing normalized variables $\zeta_t=(\varepsilon_t^2-\lambda_1)/\sigma$, (3.1) can be rewritten as the bilinear equation (1.1), with $a=\sigma c,\ a_j=\sigma c_j,\ b=\lambda_1 c,\ b_j=\lambda_1 c_j$:

$$X_t=\zeta_t\left(\sigma c+\sigma\sum_{j=1}^{\infty}c_jX_{t-j}\right)+\lambda_1 c+\lambda_1\sum_{j=1}^{\infty}c_jX_{t-j}. \tag{3.5}$$

The corresponding generating functions $A(z),B(z)$ are given by

$$A(z)=\sigma\sum_{j=1}^{\infty}c_jz^j,\quad B(z)=\lambda_1\sum_{j=1}^{\infty}c_jz^j. \tag{3.6}$$

As in Section 2, put $G(z)=(1-B(z))^{-1}=\sum_{j=0}^{\infty}g_jz^j$ and $H(z)=A(z)G(z)=\sum_{j=1}^{\infty}h_jz^j$. Note that $\bar b=\sum_{j=1}^{\infty}b_j=\lambda_1\sum_{j=1}^{\infty}c_j$ and $\mu=c\lambda_1/(1-\bar b)$. Also note $h_j=(\sigma/\lambda_1)g_j\ (j\geqslant 1)$

and

$$g_j = \sum_{k=1}^{j} \lambda_1^k \sum_{0=i_0<i_1<\cdots<i_{k-1}} c_{i_1} c_{i_2-i_1} \ldots c_{i_{k-2}-i_{k-1}} c_{j-i_{k-1}} \quad (j \geqslant 1), \quad g_0 = 1. \tag{3.7}$$

Theorem 3.1. *Let* $0 < \sigma < \infty$. *Then a covariance stationary solution of* (3.1) *exists if and only if* $\bar{b} < 1$ *and* $\|h\| < 1$ *hold, in which case the above solution is unique, ergodic and is given by the convergent orthogonal Volterra series*:

$$X_t = \mu + \mu(\sigma/\lambda_1)\zeta_t \sum_{k=1}^{\infty} \sum_{s_k<\cdots<s_1<t} h_{t-s_1} \ldots h_{s_{k-1}-s_k} \zeta_{s_1} \ldots \zeta_{s_k}$$

$$+ \mu \sum_{k=1}^{\infty} \sum_{s_k<\cdots<s_1<t} h_{t-s_1} \ldots h_{s_{k-1}-s_k} \zeta_{s_1} \ldots \zeta_{s_k}. \tag{3.8}$$

Moreover, $\mathrm{cov}(X_0, X_t) \geqslant 0$ *and*

$$\mathrm{cov}(X_t, X_0) = \frac{(a + \mu\bar{a})^2}{1 - \|h\|} \sum_{j=0}^{\infty} g_j g_{j+t},$$

$$a + \mu\bar{a} = c\left(\sqrt{\lambda_2 - \lambda_1^2} + \frac{\lambda_1^2 \sum_1^{\infty} c_j}{1 - \lambda_1 \sum_1^{\infty} c_j}\right). \tag{3.9}$$

Proof. Let $\bar{b} < 1$ and $\|h\| < 1$. Then the assumptions of Proposition 2.7(ii) applies and the existence of the solution together with (3.8) follows from Proposition 2.7 and Theorem 2.4.

Conversely, assume that a covariance stationary solution $\{X_t\}$ of (3.1) exists. As $\bar{b} < 1$ is a necessary condition for its existence, it remains to show the necessity of $\|h\| < 1$. Put $\bar{X}_t = X_t - \mu$. By (3.5),

$$\bar{X}_t = Y_t + \sum_{j=1}^{\infty} b_j \bar{X}_{t-j}, \tag{3.10}$$

where $Y_t = \zeta_t A_t = \zeta_t(\sigma c + \sigma \sum_{j=1}^{\infty} c_j X_{t-j}) = \zeta_t(a/(1-\bar{b}) + \sum_{j=1}^{\infty} a_j \bar{X}_{t-j})$ are uncorrelated and $EY_t^2 =: \sigma_Y^2 > 0$ does not depend on t by the covariance stationarity of $\{X_t\}$. By inverting (3.10), one obtains $\bar{X}_t = G(L)Y_t$ and therefore $\sum_{j=1}^{\infty} a_j \bar{X}_{t-j} = \sum_{j=1}^{\infty} h_j Y_{t-j}$. Hence $\sigma_Y^2 = EA_t^2 = (a/(1-\bar{b}))^2 + \|h\|^2 \sigma_Y^2$, thereby proving $\|h\| < 1$. The non-negativity of the covariance together with (3.9) follow from (1.15) and the non-negativity of g_j (3.7). □

Theorem 3.1 and representation (3.8) allow us to obtain further results of Giraitis et al. (2000a, Propositions 3.1 and 3.2), under the weaker condition $\|h\| < 1$ instead of (3.4).

Corollary 3.2. *Let* $\{X_t\}$ *be the* ARCH(∞) *sequence of* (3.1), *satisfying conditions* (3.2) *and* $\|h\| < 1$. *Then*

$$\sum_{t\in\mathbb{Z}} \mathrm{cov}(X_0, X_t) < \infty. \tag{3.11}$$

Assume additionally, for some constants $0 < c_- < c_+$, $\gamma > 1$, *that* $c_- j^{-\gamma} \leqslant c_j \leqslant c_+ j^{-\gamma}$, $j \geqslant 1$. *Then there are constants* $0 < C_- < C_+ < \infty$ *such that for all* $t \geqslant 1$

$$C_- t^{-\gamma} \leqslant \mathrm{cov}(X_0, X_t) \leqslant C_+ t^{-\gamma}. \tag{3.12}$$

Proof. Eq. (3.11) is immediate from Theorem 3.1, (1.15) and/or Proposition 2.7(ii), while (3.12) follows from (1.15) and

$$\tilde{c}_- j^{-\gamma} \leqslant g_j \leqslant \tilde{c}_+ j^{-\gamma} \quad (\exists 0 < \tilde{c}_- \leqslant \tilde{c}_+ < \infty) \tag{3.13}$$

and the elementary inequalities: $\alpha_- t^{-\gamma} \leqslant \sum_{j=1}^{\infty} j^{-\gamma}(t+j)^{-\gamma} \leqslant \alpha_+ t^{-\gamma}$, $t \geqslant 1$, where $0 < \alpha_- < \alpha_+ < \infty$ are some constants. The lower bound in (3.13) is immediate by $g_j \geqslant \lambda_1 c_j$, see (3.7). The upper bound seems to be well known from analysis. It can also be proved by the following argument. Write (3.7) as $g_j = \sum_{k=1}^{j} g_j^{(k)}$. It suffices to show that there exist $0 < C < \infty$ and $0 < d < 1$ such that for all $k, j \geqslant 1$

$$g_j^{(k)} \leqslant C d^k j^{-\gamma}. \tag{3.14}$$

Relation (3.14) follows from the recurrent equation $g_j^{(k)} = \lambda_1 \sum_{i=1}^{j-1} c_i g_{j-i}^{(k-1)}$, by induction on $k \geqslant 1$, see Giraitis et al. (2000b, Lemma 4.2). □

Example 3.3. Consider the classical GARCH(1,1) equations: $r_t^2 = \varepsilon_t^2 \sigma_t^2$, $\sigma_t^2 = \alpha_0 + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2$, which can be rewritten as

$$r_t^2 = \varepsilon_t^2 \left(\alpha_0/(1-\beta) + \alpha \sum_{j=1}^{\infty} \beta^{j-1} r_{t-j}^2 \right)$$

or in the form (3.1) with $c = \alpha_0/(1-\beta)$, $c_j = \alpha\beta^{j-1}$. In this case $g_j = (1-\beta)(\lambda_1\alpha + \beta)^{j-1}$, $h_j = \sigma\alpha(\lambda_1\alpha+\beta)^{j-1}$. Inequalities $\bar{b} < 1$ and $\|h\| < 1$ are equivalent to $\alpha\lambda_1 + \beta < 1$ and

$$\alpha^2\lambda_2 + 2\alpha\beta\lambda_1 + \beta^2 < 1, \tag{3.15}$$

respectively. Condition (3.15) for the existence of the covariance stationary solution $\{r_t^2\}$ of GARCH(1,1) was obtained in Karanasos (1999), He and Teräsvirta (1999), Kazakevičius et al. (2001).

From (3.8) we obtain the following orthogonal Volterra representation of GARCH(1,1):

$$r_t^2 = \mu + \mu_1 \zeta_t \sum_{k=1}^{\infty} (\alpha\sigma/\gamma)^k \sum_{s_k < \cdots < s_1 < t} \gamma^{t-s_k} \zeta_{s_1} \ldots \zeta_{s_k}$$
$$+ \mu_2 \sum_{k=1}^{\infty} (\alpha\sigma/\gamma)^{k-1} \sum_{s_k < \cdots < s_1 < t} \gamma^{t-s_k} \zeta_{s_1} \ldots \zeta_{s_k},$$

where $\zeta_t = (\varepsilon_t^2 - \lambda_1)/\sigma$ and $\gamma := \alpha\lambda_1 + \beta$, $\mu := \alpha_0\lambda_1/(1-\gamma)$, $\mu_1 := \alpha_0\sigma/(1-\beta)$, $\mu_2 := \alpha_0\sigma/\gamma$.

4. Long memory in conditional mean and conditional variance

In this section we discuss some concrete examples of generating functions $A(z)$ and $B(z)$ satisfying Assumptions A_1 and A_2 which allow us to model long memory in conditional mean and/or conditional variance.

Recall that a *weak white noise* is a sequence $\{v_t,\ t\in\mathbb{Z}\}$ of random variables with zero mean and covariance $\mathrm{cov}(v_t,v_s)=\delta_{t-s}$. Let $\{W_t,\ t\in\mathbb{Z}\}$ be a 2nd order process having moving average representation:

$$W_t = m + \sum_{j=0}^{\infty} p_j v_{t-j}, \tag{4.1}$$

where $m\in\mathbb{R}$ is a constant, $\{v_t,\ t\in\mathbb{Z}\}$ is a weak white noise, and $\sum_{j=1}^{\infty} p_j^2 < \infty$.

Definition 4.1. The process $\{W_t\}$ (4.1) will be called long memory with fractional parameter $0<d<\frac{1}{2}$ if

$$p_j \sim c j^{d-1}, \quad j\to\infty,$$

where $c\neq 0$. $\{W_t\}$ (4.1) will be called short memory if $\sum_{j=0}^{\infty}|p_j|<\infty$.

It is well-known that a moving average representation (4.1) exists if and only if the spectral density f of $\{W_t\}$ satisfies $\int_{-\pi}^{\pi}\log f(\lambda)\,\mathrm{d}\lambda > -\infty$. Note that $\{W_t\}$ being short memory implies that its covariance function $\mathrm{cov}(W_0,W_t)=Ev_0^2(p\star p)_t$ is absolutely summable: $\sum_{t\in\mathbb{Z}}|\mathrm{cov}(W_0,W_t)|<\infty$. On the other hand, if $\{W_t\}$ is long memory with fractional parameter $0<d<\frac{1}{2}$ then the covariance is not summable and

$$\mathrm{cov}(W_0,W_t)\sim c_0 t^{2d-1}, \quad t\to\infty,$$

where $c_0=c^2Ev_0^2B(d,1-2d)$ and $B(\cdot,\cdot)$ is the beta-function.

Definition 4.2. Let X_t, A_t, B_t be defined as in (1.1) and (1.2). We say that $\{X_t\}$ *exhibits long memory in conditional mean* if $\{B_t\}$ is a long-memory process with fractional parameter $0<d_1<\frac{1}{2}$. Similarly, we say that $\{X_t\}$ *exhibits long memory in conditional variance* if $\{A_t\}$ is a long-memory process with fractional parameter $0<d_2<\frac{1}{2}$.

Under conditions of Theorem 2.2, the processes $\{A_t\}$ and $\{B_t\}$ admit moving average representations (1.16) with respect to the weak white noise $\{v_s=Y_s=\zeta_sA_s\}$. Thus, $\{X_t\}$ exhibits long memory in conditional mean with fractional parameter $0<d_1<\frac{1}{2}$ if and only if

$$g_j\sim c_1 j^{d_1-1}, \quad j\to\infty \quad (\exists c_1\neq 0). \tag{4.2}$$

In view of the last equality of (1.16), condition (4.2) is also necessary and sufficient in order that $\{X_t\}$ itself is long memory with fractional parameter d_1. Similarly, $\{X_t\}$ exhibits long memory in conditional variance with fractional parameter $0<d_2<\frac{1}{2}$ if and only if

$$h_j\sim c_2 j^{d_2-1}, \quad j\to\infty \quad (\exists c_2\neq 0). \tag{4.3}$$

Proposition 4.3. *Assume the generating functions* $A(z)$ *and* $B(z)$ *are given by*

$$1 - B(z) = P_1(z)(1-z)^{d_1}, \qquad A(z) = P_2(z)(1-z)^{d_1-d_2}, \tag{4.4}$$

where $0 \leqslant d_i < \frac{1}{2}$ *and* $P_i(z) = \sum_{j=0}^{\infty} p_{ij} z^j$, $i = 1,2$ *satisfy the following conditions*:

(i) $P_1(z)$ *has no zeros in* $\{|z| \leqslant 1\}$ *and* $\sum_{j=0}^{\infty} j^2 |p_{1j}| < \infty$;

(ii) $P_2(1) \neq 0$, $\sum_{j=0}^{\infty} |p_{2j}| < \infty$ *and* $p_{2j} = o(j^{-1})$.

Then Assumptions A_1 *and* A_2 *with exception of* (1.8) *are satisfied. Moreover, if* $d_1, d_2 > 0$, *then* $\{g_j\}$ *and* $\{h_j\}$ *satisfy* (4.2) *and* (4.3), *respectively, with the asymptotic constants*

$$c_1 = (P_1(1)\Gamma(d_1))^{-1}, \quad c_2 = P_2(1)(P_1(1)\Gamma(d_2))^{-1}.$$

Furthermore, $\sum_{j=0}^{\infty} |g_j| < \infty$ *if* $d_1 = 0$, *and* $\sum_{j=1}^{\infty} |h_j| < \infty$ *if* $d_2 = 0$.

Proof. First note that if $P_1(z)$ satisfies (i) then $Q(z) := P_1^{-1}(z) = \sum_{j=0}^{\infty} q_j z_j$ satisfies (ii). Indeed, by (i), $Q(1) = P_1^{-1}(1) \neq 0$ and the function $v(x) := \sum_{j=0}^{\infty} p_{1j} e^{ijx}$ has a bounded second derivative: $|(d^2/dx^2)v(x)| \leqslant \sum_{j=0}^{\infty} |p_{1j}| j^2$. Therefore $v^{-1}(x) = \sum_{j=0}^{\infty} q_j e^{ijx}$ also has a bounded second derivative. The last fact implies $\sum_{j=0}^{\infty} |q_j j| < \infty$ and therefore $Q(z)$ satisfies (ii). Next, observe that if $Q_1(z), Q_2(z)$ satisfy (ii) then the product $Q_1(z)Q_2(z)$ satisfies (ii) as well.

To check (4.2), note that if $|d| < 1$, $d \neq 0$, then $(1-z)^{-d} = 1 + \sum_{j=1}^{\infty} \pi_j z_j$, where

$$\pi_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} \sim \frac{1}{\Gamma(d)} j^{-1+d}.$$

Therefore $(1 - B(z))^{-1} = P_1^{-1}(z)(1-z)^{-d_1} = \sum_{j=0}^{\infty} (q * \pi)_j z^j$. Since $\sum_{j=0}^{\infty} |q_j j| < \infty$, it is easy to show that

$$g_j \equiv (q*\pi)_j = \sum_{k=0}^{j} q_{j-k}\pi_k \sim \left(\sum_{k=0}^{\infty} q_k\right)\pi_j \sim (P_1(1)\Gamma(d_1))^{-1} j^{d_1-1}.$$

Thus (4.2) holds. Using a similar argument, (4.3) follows from $H(z) = A(z)(1-B(z))^{-1} = (P_2(z)/P_1(z))(1-z)^{-d_2} = \sum_{j=0}^{\infty} h_j z^j$, where

$$h_j \sim P_2(1)(P_1(1)\Gamma(d_2))^{-1} j^{d_2-1}.$$

This proves the validity of Assumption A_1 with exception of (1.8). Assumption A_2 can be easily checked using (4.2), (4.3) and the dominated convergence theorem, since $|a_j^{(n)}| \leqslant |a_j| \leqslant C|j|^{-1+d_2-d_1}$, $|b_j^{(n)}| \leqslant |b_j| \leqslant C|j|^{-1-d_1}$, where $0 < d_1, d_2 < \frac{1}{2}$. □

Corollary 4.4. *Under the assumptions of Proposition* 4.3, $\{X_t\}$ *exhibits*:

(i) *long memory in conditional mean and in conditional variance if* $0 < d_1, d_2 < \frac{1}{2}$;

(ii) *long memory in conditional mean and short memory in conditional variance if* $0 < d_1 < \frac{1}{2}$, $d_2 = 0$;

(iii) *short memory in conditional mean and long memory in conditional variance if* $d_1 = 0$, $0 < d_2 < \frac{1}{2}$;

(iv) *short memory in conditional mean and in conditional variance if* $d_1 = 0$, $d_2 = 0$.

5. Hyperbolic decay of covariance functions

Consider the bilinear model (1.1) with $b=0$ which exhibits long memory both in conditional mean and in conditional variance in the sense of Definition 4.2, or (4.2)–(4.3). In this section, we study the implications of (4.2) and (4.3) on the decay rate of the autocovariance functions of the "observable" sequences $\{X_t\}$ and $\{X_t^2\}$ as well as of the cross-covariance between the two sequences; in other words, the asymptotics of $\mathrm{cov}(X_s,X_t), \mathrm{cov}(X_s,X_t^2), \mathrm{cov}(X_s^2,X_t^2)$ as $|t-s|\to\infty$. Put $\mu_p := E\zeta_0^p$.

Assumption A₄. The fourth moment $\mu_4 = E\zeta_0^4 < \infty$ and

$$11\mu_4^{1/2}\|h\|^2 < 1.$$

The above moment assumption was introduced in Giraitis et al. (2000b). As shown in the last paper, Assumption A_4 guarantees the existence of finite fourth moment EY_0^4, where $Y_t = \zeta_t A_t$ is the stationary solution to the LARCH equation

$$Y_t = \zeta_t\left(a + \sum_{j=1}^{\infty} h_j Y_{t-j}\right), \tag{5.1}$$

see Section 1.

Lemma 5.1 (Giraitis et al., 2001). *Let Assumption* A_4 *and* (4.3) *be satisfied. Then*

$$\mathrm{cov}(Y_0, Y_t^2) \sim c_3 t^{d_2-1}, \quad t\to\infty, \tag{5.2}$$

where $c_3 := 2a^3c_2/(1-\|h\|^2)^2$.

Remark 5.2. The moment assumption A_4 for the existence of EY_t^4 was improved in Giraitis et al. (2001) to

$$\mu_4\|h\|_4^4 + 4|\mu_3|\,\|h\|_3^3 + 6\|h\|^2 < 1.$$

The last paper also obtains conditions for $E|Y_t|^3 < \infty$ and (5.2) which do not require finiteness of μ_4.

Lemma 5.3. *Let Assumption* A_4 *and* (4.3) *be satisfied. Then there exists a finite constant* C *such that for any integers* $t'' < t' \leqslant t$

$$|EY_{t''}Y_{t'}Y_t^2| \leqslant C|t-t'|_+^{d_2-1}|t'-t''|^{d_2-1}. \tag{5.3}$$

The proof of Lemma 5.3 is given at the end of Section 5. Define the following asymptotic constants:

$$\kappa_{11} = (ac_1/(1-\|h\|^2))^2 B(d_1, 1-2d_1),$$

$$\kappa_{12}^{+} = \frac{2a^3c_1c_2\|g\|^2}{(1-\|h\|^2)^2} B(d_1, 1-d_1-d_2),$$

$$\kappa_{12}^{-} = \frac{2a^3 c_1 c_2 \|g\|^2}{(1-\|h\|^2)^2} B(d_2, 1-d_1-d_2),$$

$$\kappa_{22}' = 2(a^2 c_1^2/(1-\|h\|^2))^2 B^2(d_1, 1-2d_1),$$

$$\kappa_{22}'' = (4a^4 c_2^2 \|g\|^4/(1-\|h\|^2)^3) B(d_2, 1-2d_2).$$

Theorem 5.4. *Let Assumptions* A_1 *and* A_2 *as well as* (4.2) *and* (4.3) *be satisfied, with* $0 < d_1 < \frac{1}{2}$, $0 < d_2 < \frac{1}{2}$. *Then*:

(i)

$$\operatorname{cov}(X_0, X_t) \sim \kappa_{11} |t|^{2d_1 - 1}, \quad |t| \to \infty;$$

(ii)

$$\operatorname{cov}(X_0, X_t^2) \sim \begin{cases} \kappa_{12}^{+} t^{d_1+d_2-1}, & t \to \infty, \\ \kappa_{12}^{-} |t|^{d_1+d_2-1}, & t \to -\infty, \end{cases}$$

provided Assumption A_4 *holds*;

(iii)

$$\operatorname{cov}(X_0^2, X_t^2) \sim \begin{cases} \kappa_{22}' |t|^{2(2d_1-1)} & \text{if } 2(1-2d_1) < 1-2d_2, \\ \kappa_{22}'' |t|^{2d_2-1} & \text{if } 2(1-2d_1) > 1-2d_2, \end{cases} \quad |t| \to \infty,$$

provided Assumption A_4 *holds.*

Proof. (i) Follows immediately from (4.2) and the white noise representation

$$X_t = \sum_{j=0}^{\infty} g_j Y_{t-j}, \tag{5.4}$$

see (1.16).

(ii) Let $t > 0$. Using (5.2) and Assumption A_4 one obtains

$$\begin{aligned} \operatorname{cov}(X_0, X_t^2) &= \operatorname{cov}\left(\sum_{s \leqslant 0} g_{-s} Y_s, \left(\sum_{s \leqslant t} g_{t-s} Y_s\right)^2\right) \\ &= \sum_{s_1 \leqslant 0, s_2 \leqslant t} g_{-s_1} g_{t-s_2}^2 \operatorname{cov}(Y_{s_1}, Y_{s_2}^2) \\ &\quad + 2 \sum_{s_2 < s_1 \leqslant 0} g_{t-s_1} g_{-s_1} g_{t-s_2} \operatorname{cov}(Y_{s_1}^2, Y_{s_2}). \end{aligned}$$

Put $\gamma_t := \operatorname{cov}(Y_0, Y_t^2)$. Then

$$J_t := \sum_{s_1 \leqslant 0, s_2 \leqslant t} g_{-s_1} g_{t-s_2}^2 \gamma_{s_2 - s_1} = \sum_{s_1 \leqslant 0} g_{-s_1} \sum_{u=0}^{t-s_1} g_{t-s_1-u}^2 \gamma_u \equiv \sum_{s_1 \leqslant 0} g_{-s_1} K_{t-s_1}.$$

Using Lemma 5.1, one easily obtains

$$K_t = \sum_{u=0}^{t} g_{t-u}^2 \gamma_u \sim \|g\|^2 c_3 t^{d_2-1} \quad (t \to \infty).$$

Consequently by (4.2),

$$J_t \sim c_1 c_3 \|g\|^2 \sum_{s>0} s^{d_1-1}(t+s)^{d_2-1} \sim \kappa_{12}^{+} t^{d_1+d_2-1}, \quad t \to \infty. \tag{5.5}$$

In a similar way, one can show the asymptotics $J_t \sim \kappa_{12}^{-}|t|^{d_1+d_2-1}\,(t \to -\infty)$. It remains to show

$$Q_t = \mathrm{o}(J_t), \tag{5.6}$$

where

$$Q_t = \sum_{s_2<s_1\leqslant 0\wedge t} g_{t-s_1} g_{-s_1} g_{t-s_2} \gamma_{s_1-s_2}.$$

Indeed, as $t \to \pm\infty$,

$$\begin{aligned} |Q_t| &\leqslant C \sum_{s_2<s_1\leqslant 0\wedge t} |t-s_1|_+^{d_1-1} |s_1|_+^{d_1-1} |t-s_2|_+^{d_1-1} |s_1-s_2|^{d_2-1} \\ &\leqslant C \sum_{s_1\leqslant 0} |t-s_1|^{d_1-1} |s_1|_+^{d_1-1} |t-s_1|^{d_1+d_2-1} \\ &\leqslant C t^{3d_1+d_2-2} = \mathrm{o}(t^{d_1+d_2-1}), \end{aligned}$$

proving (ii).

(iii) We have

$$X_t^2 = \sum_{s\leqslant t} g_{t-s}^2 Y_s^2 + 2 \sum_{s_2<s_1\leqslant t} g_{t-s_1} g_{t-s_2} Y_{s_1} Y_{s_2} =: x_t + y_t.$$

Here,

$$\mathrm{cov}(x_0, x_t) = \sum_{s_1\leqslant t, s_2\leqslant 0} g_{t-s_1}^2 g_{-s_2}^2 \,\mathrm{cov}(Y_{s_1}^2, Y_{s_2}^2) =: \rho_{1t}.$$

Next,

$$\begin{aligned} \mathrm{cov}(y_0, y_t) &= 4 \sum_{s_2<s_1\leqslant t} \sum_{s_4<s_3\leqslant 0} g_{t-s_1} g_{t-s_2} g_{-s_1} g_{-s_4} \,\mathrm{cov}(Y_{s_1} Y_{s_2}, Y_{s_3} Y_{s_4}) \\ &= 4 \sum_{s_2<s_1\leqslant 0} g_{t-s_1} g_{t-s_2} g_{-s_1} g_{-s_2} E Y_{s_1}^2 Y_{s_2}^2 \\ &\quad + 4 \sum_{s_2<s_1\leqslant t\wedge 0, s_3<s_1\wedge 0} g_{t-s_1} g_{t-s_2} g_{-s_1} g_{-s_3} E Y_{s_1}^2 Y_{s_2} Y_{s_3} =: \rho_{2t} + \lambda_{1t}, \end{aligned}$$

where we used the fact that $\mathrm{cov}(Y_{s_1}Y_{s_2}, Y_{s_3}Y_{s_4}) = EY_{s_1}Y_{s_2}Y_{s_3}Y_{s_4}$ vanishes for $s_1 \neq s_3$, $s_2 < s_1,\ s_4 < s_3$. Similarly,

$$\mathrm{cov}(x_0, y_t) = 2 \sum_{s_3 < s_2 \leqslant s_1 \leqslant 0} g^2_{-s_1} g_{t-s_2} g_{t-s_3} EY^2_{s_1} Y_{s_2} Y_{s_3} =: \lambda_{2t},$$

$$\mathrm{cov}(y_0, x_t) = 2 \sum_{s_2 < s_1 \leqslant s_3 \wedge 0, s_3 \leqslant t} g_{-s_1} g_{-s_2} g^2_{t-s_3} EY_{s_1} Y_{s_2} Y^2_{s_3} =: \lambda_{3t}.$$

Hence

$$\rho_t := \mathrm{cov}(X_0^2, X_t^2) = \rho_{1t} + \rho_{2t} + \sum_{i=1}^{3} \lambda_{it}.$$

It suffices to show

$$\rho_{1t} \sim \kappa''_{22}\, t^{2d_2 - 1}, \tag{5.7}$$

$$\rho_{2t} \sim \kappa'_{22}\, t^{2(2d_1 - 1)}, \tag{5.8}$$

$$\lambda_{it} = \mathrm{o}(\max(t^{2d_2-1}, t^{2(2d_1-1)})), \quad i = 1, 2, 3. \tag{5.9}$$

Let us prove (5.9) for $i = 1$. According to (4.2) and Lemma 5.3, uniformly in $s_2 < s_1 \leqslant 0$,

$$\begin{aligned}\sum_{s_3 < s_1 \wedge 0} |g_{-s_3}|\, |EY^2_{s_1} Y_{s_2} Y_{s_3}| &\leqslant C \sum_{s_3 \leqslant s_2 < s_1 \wedge 0} |s_3|^{d_1-1} |s_1 - s_2|^{d_2-1} |s_2 - s_3|_+^{d_2-1} \\ &\quad + C \sum_{s_2 < s_3 < s_1 \wedge 0} |s_3|^{d_1-1} |s_1 - s_3|^{d_2-1} |s_2 - s_3|^{d_2-1} \\ &\leqslant C |s_1 - s_2|^{d_2-1},\end{aligned}$$

where in the last sum we used the inequality $\max(|s_1 - s_3|, |s_2 - s_3|) \geqslant |s_1 - s_2|/2$. Hence

$$\begin{aligned}|\lambda_{1t}| &\leqslant C \sum_{s_2 < s_1 \leqslant 0} |t - s_1|^{d_1-1} |t - s_2|^{d_1-1} |s_1|_+^{d_1-1} |s_1 - s_2|^{d_2-1} \\ &\leqslant C \sum_{s_1 \leqslant 0} |t - s_1|^{2d_1+d_2-2} |s_1|_+^{d_1-1} \leqslant C t^{3d_1+d_2-2}.\end{aligned}$$

As $\min(1-2d_1, 2(1-2d_2)) < [(1-2d_1)+2(1-2d_2)]/2 = 3/2 - 2d_1 - d_2 < 2 - 3d_1 - d_2$, this proves (5.9) for $i = 1$.

Similarly, by (4.2) and Lemma 5.3 and noting that $2d_1 + d_2 - 2 < d_2 - 1 - \delta$ for $\delta > 0$ sufficiently small,

$$\begin{aligned}|\lambda_{3t}| &\leqslant C \sum_{s_2 < s_1 \leqslant s_3 \wedge 0, s_3 \leqslant t} |s_1|_+^{d_1-1} |s_2|_+^{d_1-1} |s_1 - s_2|_+^{d_2-1} |s_1 - s_3|_+^{d_2-1} g^2_{t-s_3} \\ &\leqslant C \sum_{s_1 \leqslant s_3 \wedge 0, s_3 \leqslant t} |s_1|_+^{2d_1+d_2-2} |s_1 - s_3|_+^{d_2-1} g^2_{t-s_3}\end{aligned}$$

$$\leqslant C \sum_{s_1 \leqslant s_3 \wedge 0, s_3 \leqslant t} |s_1|_+^{d_2-1-\delta} |s_1 - s_3|_+^{d_2-1} g_{t-s_3}^2$$

$$\leqslant C \sum_{s_3} |s_3|_+^{2d_2-1-\delta} g_{t-s_3}^2 \leqslant C t^{2d_2-1-\delta},$$

proving (5.9) for $i=3$. The proof of (5.9) for $i=2$ is analogous as in the case $i=3$.

Consider (5.7). According to Giraitis et al. (2000b), as $t \to \infty$,

$$r_t := \operatorname{cov}(Y_0^2, Y_t^2) \sim c_4 t^{2d_2-1}, \tag{5.10}$$

where $c_4 = 4a^4 c_2^2 (1 - \|h\|^2)^{-3} B(d_2, 1-2d_2)$. Note that if $\sum_{s \geqslant 0} |p_s| < \infty$ and, as $s \to \infty$, $p_s = \mathrm{o}(1/s), v_s \sim cs^{-\gamma}$, where $0 < \gamma < 1$, then

$$\sum_{s \geqslant 0} p_s v_{t+s} \sim v_t \sum_{s \geqslant 0} p_s, \quad t \to \infty. \tag{5.11}$$

Applying (5.10) and (5.11) to $\rho_{1t} = \sum_{s_1,s_2 \geqslant 0} g_{s_1}^2 g_{s_2}^2 r_{t-s_1+s_2}$, one obtains $\rho_{1t} \sim (\sum_{s \geqslant 0} g_s^2)^2 r_t$, or (5.7).

It remains to show (5.8). Write

$$\rho_{2t} = 4(EY_0^2)^2 \sum_{s_2 < s_1 \leqslant 0} g_{t-s_1} g_{-s_1} g_{t-s_2} g_{-s_2}$$

$$+ 4 \sum_{s_2 < s_1 \leqslant 0} g_{t-s_1} g_{-s_1} g_{t-s_2} g_{-s_2} \operatorname{cov}(Y_{s_1}^2, Y_{s_2}^2) =: \rho_{2t}' + \rho_{2t}''.$$

We have

$$\rho_{2t}' = 2(EY_0^2)^2 \left[\left(\sum_{s \geqslant 0} g_{t+s} g_s \right)^2 - \sum_{s \geqslant 0} g_{t+s}^2 g_s^2 \right] \sim \kappa_{22}' t^{2(2d_1-1)},$$

while $\rho_{2t}'' = \mathrm{o}(t^{2(2d_2-1)})$ easily follows. This proves (5.8) and the theorem. □

Proof of Lemma 5.3. We shall use the results and notation of the paper Giraitis et al. (2000b).

We use the representation $Y_t = \zeta_t A_t$, with $A_t = a + A_t^0$ given by (1.10). Let $t > 0$, then

$$A_t = \sum_{\tau=0}^{t-1} \mathbf{A}_{0,\tau} A_{[\tau,t)}, \tag{5.12}$$

where, for any integers $u \leqslant \tau < t$,

$$\mathbf{A}_{u,\tau} := a \sum_{k=0}^{\infty} \sum_{s_k < \cdots < s_1 < u} h_{\tau-s_1} \ldots h_{s_{k-1}-s_k} \zeta_{s_1} \ldots \zeta_{s_k},$$

$$A_{[\tau,t)} := \zeta_\tau \sum_{k=0}^{t-1} \sum_{\tau < s_{k-1} < \cdots < s_1 < t} h_{t-s_1} \ldots h_{s_{k-1}-\tau} \zeta_{s_1} \ldots \zeta_{s_{k-1}}.$$

Consider first the case $t = t' > t'' = 0$. Then $EY_t^3 Y_0 = \mu_3 EA_t^3 Y_0$ and by (5.12) we obtain

$$EA_t^3 Y_0 = EA_t^3 \zeta_0 A_0 = E\left(\sum_{\tau=0}^{t-1} \mathbf{A}_{0,\tau} A_{[\tau,t)}\right)^3 \zeta_0 A_0$$

$$= \sum_{\tau_1,\tau_2,\tau_3=0}^{t-1} E[A_{[\tau_1,t)} A_{[\tau_2,t)} A_{[\tau_3,t)} \zeta_0] E[A_0 \mathbf{A}_{0,\tau_1} \mathbf{A}_{0,\tau_2} \mathbf{A}_{0,\tau_3}].$$

Note $EA_0^4 < \infty$ and $\sup_{\tau>0} E\mathbf{A}_{0,\tau}^4 < \infty$, which follows from Assumption A4 and (Giraitis et al., 2000b, Lemma 3.1). Hence $|EA_0 \mathbf{A}_{0,\tau_1} \mathbf{A}_{0,\tau_2} \mathbf{A}_{0,\tau_3}| \leqslant C < \infty$ and the statement of the lemma for $0 = t'' < t' = t$ follows from

$$\sum_{\tau_1,\tau_2,\tau_3=0}^{t-1} |EA_{[\tau_1,t)} A_{[\tau_2,t)} A_{[\tau_3,t)} \zeta_0| \leqslant Ct^{d_2-1}. \tag{5.13}$$

The last expectation can be written as

$$EA_{[\tau_1,t)} A_{[\tau_2,t)} A_{[\tau_3,t)} \zeta_0 = \sum_{(S)_3} h_t^{S_1} h_t^{S_2} h_t^{S_3} E[\zeta^{S_1} \zeta^{S_2} \zeta^{S_3} \zeta_0] 1\,(\wedge S_i = \tau_i, i = 1,2,3), \tag{5.14}$$

where the sum is taken over all collections $(S)_3 = (S_1, S_2, S_3)$ of non-empty ordered subsets $S_i \subset [\tau_i, t) \cap \mathbb{Z}$ with $\bigwedge S_i := \min(s\colon s \in S_i) = \tau_i$, $i = 1,2,3$, and where, for any such ordered subset $S = \{s_k, \ldots, s_1\}$, $s_k < \cdots < s_1$,

$$h_t^S := h_{t-s_1} h_{s_1-s_2} \ldots h_{s_{k-1}-s_k}, \quad \zeta^S := \zeta_{s_1} \ldots \zeta_{s_k}.$$

Using the diagram argument of (Giraitis et al., 2000b, Lemmas 3.2 and 4.2), one can show the bounds

$$\sum_{(S)_3} |h_t^{S_1} h_t^{S_2} h_t^{S_3}|\, |E\zeta^{S_1} \zeta^{S_2} \zeta^{S_3}|\, 1\,(\wedge(S_1 \cup S_2 \cup S_3) = \tau) \leqslant C|t-\tau|^{-2(1-d_2)}, \tag{5.15}$$

and

$$\sum_{\tau_1,\tau_2,\tau_3=\tau}^{t} |EA_{[\tau_1,t)} A_{[\tau_2,t)} A_{[\tau_3,t)}| \leqslant C|t-\tau|^{-2(1-d_2)}, \tag{5.16}$$

where $2(1-d_2) > 1$, which will be used to prove (5.13). Without loss of generality, one may assume $\mu_3 = E\zeta_0^3 \neq 0$.

Note $E[\zeta^{S_1} \zeta^{S_2} \zeta^{S_3} \zeta_0] = 0$ in (6.14) if $\bigwedge(S_1 \cup S_2 \cup S_3) = \tau_1 \wedge \tau_2 \wedge \tau_3 > 0$. Let $\tau_3 = \bigwedge S_3 = 0$. There are two cases: (1) $\bigwedge(S_1 \cup S_2) = \tau_1 \wedge \tau_2 = 0$ and (2) $\tau_1, \tau_2 > 0$. In case (1), $|E\zeta^{S_1} \zeta^{S_2} \zeta^{S_3} \zeta_0| \leqslant C|E\zeta^{S_1} \zeta^{S_2} \zeta^{S_3}|$ and the sum on the r.h.s. of (5.14) over such $(S)_3$ can be bounded by the r.h.s. of (5.15) which is $o(t^{d_2-1})$.

In case (2), or $\tau_1 \wedge \tau_2 > 1, \tau_3 = 0$, from the identity $A_{[0,t)} = \zeta_0 \sum_{u=0}^{t-1} h_u A_{[u,t)}$ one has $EA_{[\tau_1,t)} A_{[\tau_2,t)} A_{[0,t)} \zeta_0 = E\zeta_0^2 \sum_{u=0}^{t-1} h_u EA_{[\tau_1,t)} A_{[\tau_2,t)} A_{[u,t)}$. Hence and from (5.14), (5.16)

$$\sum_{\tau_1,\tau_2=1}^{t-1} |EA_{[\tau_1,t)} A_{[\tau_2,t)} A_{[0,t)} \zeta_0| \leqslant C \sum_{u=0}^{t-1} |h_u|\, |EA_{[\tau_1,t)} A_{[\tau_2,t)} A_{[u,t)}|$$

$$\leqslant C \sum_{u=0}^{t-1} |u|_+^{d_2-1} |t-u|^{-2(1-d_2)} \leqslant Ct^{d_2-1},$$

which proves (5.13), or the bound $|EA_t^3 Y_0| \leqslant Ct^{d_2-1}$. In a similar way, one can show the more general bound: for any $s_1, s_2 \geqslant t > 0$,

$$|E\mathbf{A}_{t,s_1}\mathbf{A}_{t,s_2}A_t Y_0| \leqslant Ct^{d_2-1}. \tag{5.17}$$

Consider now the general case $0 = t'' < t' < t$. Then by (5.12)

$$\begin{aligned} EY_t^2 Y_{t'} Y_0 &= E\left(\sum_{s=t'}^{t-1} \mathbf{A}_{t',s} A_{[s,t)}\right)^2 \zeta_{t'} A_{t'} Y_0 \\ &= \sum_{s_1,s_2=t'}^{t-1} E[A_{[s_1,t)}A_{[s_2,t)}\zeta_{t'}]E[\mathbf{A}_{t',s_1}\mathbf{A}_{t',s_2}A_{t'}Y_0]. \end{aligned}$$

Whence and from (5.17), we have $|EY_t^2 Y_{t'} Y_0| \leqslant C|t'|^{d_2-1} \sum_{s_1,s_2=t'}^{t-1} |EA_{[s_1,t)}A_{[s_2,t)}\zeta_{t'}|$, and the lemma follows from the bound

$$\sum_{s_1,s_2=t'}^{t-1} |EA_{[s_1,t)}A_{[s_2,t)}\zeta_{t'}| \leqslant C|t-t'|^{d_2-1},$$

whose proof is similar (actually, simpler) to that of (5.13). □

6. Convergence of partial sums' processes

In this section we consider the weak convergence of partial sums' processes $\{S_{N1}(\tau), 0 \leqslant \tau \leqslant 1\}$ and $\{S_{N2}(\tau), 0 \leqslant \tau \leqslant 1\}$, where

$$S_{N1}(\tau) := \sum_{s=1}^{[N\tau]} X_s, \quad S_{N2}(\tau) := \sum_{s=1}^{[N\tau]} (X_s^2 - EX_s^2)$$

and where $\{X_t\}$ is the stationary solution of (1.1) with $b = 0$ which exhibits long memory in conditional mean and in conditional variance in the sense of Definition 4.2.

Let us introduce the fractional Brownian motion, $J_1(\tau; d)$, and the Rosenblatt process, $J_2(\tau; d)$, as the stochastic integrals

$$J_1(\tau; d) = k_1(d) \int_{\mathbb{R}} \left[\int_0^\tau (s-x)_+^{d-1}\, \mathrm{d}s\right] W(\mathrm{d}x), \tag{6.1}$$

$$J_2(\tau; d) = k_2(d) \int_{\mathbb{R}^2} \left[\int_0^\tau (s-x_1)_+^{d-1}(s-x_2)_+^{d-1}\, \mathrm{d}s\right] W(\mathrm{d}x_1)W(\mathrm{d}x_2) \tag{6.2}$$

with respect to standard Gaussian white noise $W(\mathrm{d}x)$ with zero mean and variance $\mathrm{d}x$; see e.g. Taqqu (1975). The normalizations $k_i(d)$, $i = 1, 2$ in (6.1), (6.2) are chosen so that $EJ_i^2(1; d) = 1$, $i = 1, 2$. The processes $J_1(\tau; d)$ and $J_2(\tau; d)$ are defined for $0 < d < \frac{1}{2}$ and $\frac{1}{4} < d < \frac{1}{2}$, respectively. Recall that the fractional Brownian motion can be alternatively defined for any $d \in (-\frac{1}{2}, \frac{1}{2})$ as the Gaussian process with zero mean and the covariance

$$EJ_1(\tau;d)J_1(\tau';d) = (1/2)(|\tau|^{1+2d} + |\tau'|^{1+2d} - |\tau-\tau'|^{1+2d}).$$

Write $\Rightarrow_{D[0,1]}$ and $\Rightarrow$ for the weak convergence of random elements in the Skorohod space $D[0,1]$ and the convergence of finite dimensional distributions, respectively.

Consider first the partial sums' process $\{S_{N2}(\tau)\}$. Put $S_{Ni}=S_{Ni}(1)$, $i=1,2$. According to Theorem 5.4(iii), under Assumption A_4,

$$\operatorname{var} S_{N2} \sim \begin{cases} \tilde{\kappa}'_{22} N^{1+2d_2} & \text{if } 2(1-2d_1) > 1-2d_2, \\ \tilde{\kappa}''_{22} N^{4d_1} & \text{if } 2(1-2d_1) < 1-2d_2, \end{cases} \tag{6.3}$$

where $\tilde{\kappa}'_{22} = \kappa'_{22}/(d_2(1+2d_2))$, $\tilde{\kappa}''_{22} = \kappa''_{22}/(2d_1(4d_1-1))$.

Theorem 6.1. *Let conditions of Theorem* 5.4(iii) *be satisfied. Then*

$$(\operatorname{var} S_{N2})^{-1/2} S_{N2}(\tau) \quad \Rightarrow_{D[0,1]} \quad \begin{cases} J_1(\tau;d_2) & \textit{if } 1-2d_2 < 2(1-2d_1), \\ J_2(\tau;d_1) & \textit{if } 1-2d_2 > 2(1-2d_1). \end{cases} \tag{6.4}$$

Proof. The tightness of random elements $\{(\operatorname{var} S_{N2})^{-1/2} S_{N2}(\tau), \tau\in[0,1]\}$ in $D[0,1]$ follows from (6.3) and the stationarity of increments, as $E(S_{N2}(\tau') - S_{N2}(\tau))^2 = ES^2_{N2}(\tau'-\tau) = \operatorname{var} S_{[N(\tau'-\tau)]2} \leqslant C(\operatorname{var} S_{N2})|\tau'-\tau|^{\gamma}$, $0\leqslant \tau < \tau' \leqslant 1$, with $\gamma > 1$; see e.g. Billingsley (1968).

It remains to prove the finite dimensional convergence. We shall prove the convergence of one-dimensional distributions at $\tau=1$ only, as the general case can be treated analogously. With (6.1)–(6.2) in mind, write $S_{N2} = U_{N1} + U_{N2}$, where

$$U_{N1} = 2\sum_{t=1}^{N} \sum_{s_2 < s_1 \leqslant t} g_{t-s_1} g_{t-s_2} Y_{s_1} Y_{s_2},$$

$$U_{N2} = \sum_{t=1}^{N} \sum_{s\leqslant t} g^2_{t-s}(Y^2_s - EY^2_s).$$

Let first $1-2d_2 < 2(1-2d_1)$. Then from the proof of Theorem 5.4(iii) one obtains $\operatorname{var} U_{N2} \sim \operatorname{var} S_{N2}$, $\operatorname{var} U_{N1} = o(\operatorname{var} S_{N2})$, and the convergence in distribution $(\operatorname{var}(S_{N2}))^{-1/2} S_{N2} \Rightarrow J_1(1;d_2)$ follows from

$$N^{-1/2-d_2} U_{N2} \quad \Rightarrow \quad \alpha J_1(1;d_2), \tag{6.5}$$

where $\alpha = (\tilde{\kappa}'_{22})^{1/2} = 2ac_2\|g\|^2/\{(1-\|h\|^2)(d_2(1+2d_2))^{1/2}\}$.

Given a large $K>0$, split $U_{N2} = U^-_{N2} + U^+_{N2}$, where

$$U^-_{N2} := \sum_{t=1}^{N} \sum_{t-K<s\leqslant t} g^2_{t-s}(Y^2_s - EY^2_s), \quad U^+_{N2} := \sum_{t=1}^{N} \sum_{s\leqslant t-K} g^2_{t-s}(Y^2_s - EY^2_s).$$

It suffices to show that, for all $N\geqslant 1$

$$E(U^+_{N2})^2 \leqslant \delta_K N^{2d_2+1}, \tag{6.6}$$

where $\delta_K \to 0$ $(K \to \infty)$, and that, for any $K < \infty$ fixed,

$$N^{-1/2-d_2} U_{N2}^{-} \;\Rightarrow\; \alpha_K J_1(1; d_2), \tag{6.7}$$

where $\alpha_K \to \alpha\,(K \to \infty)$. Here, (6.6) follows from (5.10) while (6.7) follows from

$$N^{-1/2-d_2} \sum_{t=1}^{N} (Y_t^2 - EY_t^2) \;\Rightarrow\; ((\tilde{\kappa}_{22}')^{1/2}/\|g\|^2) J_1(1; d_2),$$

see Giraitis et al. (2000b, Theorem 2.3), since U_{N2}^{-} can be represented as $U_{N2}^{-} = \sum_{i=0}^{K-1} g_i^2 \sum_{t=1}^{N} (Y_t^2 - EY_t^2) + O_P(1)$. This proves (6.5) and the first part of the theorem, too.

Next, consider the case $1-2d_2 > 2(1-2d_1)$. In this case from the proof of Theorem 5.4(iii) one obtains $\operatorname{var} U_{N1} \sim \operatorname{var} S_{N2}$, $\operatorname{var} U_{N2} = o(\operatorname{var} S_{N2})$, and $(\operatorname{var}(S_{N2}))^{-1/2} S_{N2} \Rightarrow J_2(1; d_1)$ follows from

$$N^{-2d_1} U_{N1} \;\Rightarrow\; (\tilde{\kappa}_{22}'')^{1/2} J_2(1; d_1). \tag{6.8}$$

The proof of (6.8) uses the "scheme of discrete multiple integrals" (Surgailis, 1982; Surgailis and Vaičiulis, 1999). Consider a quadratic form

$$Q(\phi) = \sum_{s_1 \neq s_2} \phi(s_1, s_2) Y_{s_1} Y_{s_2} \tag{6.9}$$

in martingale differences Y_s (5.1). (Below, we use the fact that the sequence $\{Y_s\}$ is ergodic, which follows from Theorem 2.2 applied to the bilinear equation (5.1), or from the ergodicity of the sequence $\{A_t = a + A_t^0\}$ given by Volterra series (1.10).) We claim that there exists a constant $C < \infty$ independent of ϕ and such that

$$EQ^2(\phi) \leqslant C\|\phi\|^2, \tag{6.10}$$

where $\|\phi\|^2 := \sum_{s_1 \neq s_2} \phi^2(s_1, s_2)$. It suffices to check (6.10) for ϕ symmetric: $\phi(s_1, s_2) = \phi(s_2, s_1)$ and vanishing everywhere except for a finite number of points $(s_1, s_2) \in \mathbb{Z}^2$. By the martingale property of Y_s,

$$EQ^2(\phi) = 4 \sum_{s_2 < s_1} \phi^2(s_1, s_2) E[Y_{s_1}^2 Y_{s_2}^2] + 8 \sum_{s_3 < s_2 < s_1} \phi(s_1, s_2)\phi(s_1, s_3) E[Y_{s_3} Y_{s_2} Y_{s_1}^2]$$

$$=: 4I_1 + 8I_2.$$

Here, $I_1 \leqslant C\|\phi\|^2$ as $EY_s^4 \leqslant C$. Next, by Lemma 5.3,

$$|I_2| \leqslant C \sum_{s_3 < s_2 < s_1} |\phi(s_1, s_2)\phi(s_1, s_3)|\, |s_1 - s_2|_+^{d_2-1} |s_2 - s_3|_+^{d_2-1}$$

$$\leqslant C \sum_{s_2 < s_1} |\phi(s_1, s_2)|\, |s_1 - s_2|_+^{d_2-1} \psi(s_1),$$

where $\psi(s_1) = \{\sum_{s_3} |\phi(s_1, s_3)|^2\}^{1/2}$ and where we used the fact that $\sum_{s_3} |s_2 - s_3|_+^{2(d_2-1)} \leqslant C$. Hence by the Cauchy–Schwarz inequality,

$$|I_2| \leqslant C\|\phi\| \left(\sum_{s_1, s_2} \psi^2(s_1) |s_1 - s_2|_+^{2(d_2-1)} \right)^{1/2} \leqslant C\|\phi\| \left(\sum_{s_1} \psi^2(s_1) \right)^{1/2} = C\|\phi\|^2.$$

This proves (6.10).

Introduce a stochastic measure W_N defined on intervals $(x', x'']$, $x' < x''$ by

$$W_N((x', x'']) := \sigma^{-1} N^{-1/2} \sum_{x' < s/N \leqslant x''} Y_s, \tag{6.11}$$

where $\sigma^2 = EY_0^2 = a^2/(1 - \|h\|^2)$. As $\{Y_s\}$ is an ergodic square integrable martingale difference sequence, see above, from the classical martingale CLT (Billingsley, 1968) it follows that for any $m < \infty$ and for any mutually disjoint intervals $(x_j', x_j'']$, $j = 1, \ldots, m$,

$$(W_N((x_1', x_1'']), \ldots, W_N((x_m', x_m''])) \;\Rightarrow\; (W((x_1', x_1'']), \ldots, W((x_m', x_m''])), \tag{6.12}$$

where $W(\mathrm{d}x)$ is the standard Gaussian white noise as in (6.1) and (6.2).

To show (6.8), write the l.h.s. as the "discrete double integral":

$$N^{-2d_1} U_{N1} = \int_{\mathbb{R}^2} f_N(x_1, x_2) W_N(\mathrm{d}x_1) W_N(\mathrm{d}x_2), \tag{6.13}$$

where the piecewise constant function $F_N(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$ is defined by

$$f_N(x_1, x_2) := \sigma^2 N^{1-2d_1} \sum_{t=1}^{N} g_{t-s_1} g_{t-s_2} = \sigma^2 N^{1-2d_1} \sum_{t=1}^{N} g_{t-[x_1 N]} g_{t-[x_2 N]} \tag{6.14}$$

for $(x_1, x_2) \in (s_1/N, (s_1+1)/N] \times (s_2/N, (s_2+1)/N]$ such that $s_1 \neq s_2$, $(s_1, s_2) \in \mathbb{Z}^2$, and $f_N(x_1, x_2) = 0$ elsewhere. More generally, a "discrete double integral" $\int \varphi \, \mathrm{d}^2 W_N \equiv \int_{\mathbb{R}^2} \varphi(x_1, x_2) W_N(\mathrm{d}x_1) W_N(\mathrm{d}x_2)$ is defined for each (simple) function φ taking a finite number of constant values $\varphi^{\Delta_1, \Delta_2}$ on "squares" $\Delta_1 \times \Delta_2 \subset \mathbb{R}^2$, $\Delta_i = (s_i, (s_i+1)/N]$, $s_i \in \mathbb{Z}$, $i = 1, 2$, and vanishing on "diagonals": $\varphi^{\Delta_1, \Delta_2} = 0$, $\Delta_1 = \Delta_2$, by

$$\int \varphi \, \mathrm{d}^2 W_N := \sum_{\Delta_1, \Delta_2} \varphi^{\Delta_1, \Delta_2} W_N(\Delta_1) W_N(\Delta_2)$$

with W_N given by (6.11). By definition, any such integral is an off-diagonal quadratic form of type (6.9), and bound (6.10) translates to

$$E\left(\int \varphi \, \mathrm{d}^2 W_N\right)^2 \leqslant C \|\varphi\|^2, \tag{6.15}$$

where $\|\varphi\| = (\int_{\mathbb{R}^2} \varphi^2(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2)^{1/2}$ stands for the norm in the Hilbert space $L^2(\mathbb{R}^2)$ of all real-valued square integrable functions on $\mathbb{R}^2$. Convergence (6.8) now follows from (6.12)–(6.15) and the representation

$$J_2(1; d_1) = k_2(d_1) \sigma^{-2} c_1^{-2} \int_{\mathbb{R}^2} f(x_1, x_2) W(\mathrm{d}x_1) W(\mathrm{d}x_2),$$

see (6.2), where $f(x_1, x_2) := \sigma^2 c_1^2 \int_0^1 (s - x_1)_+^{d_1 - 1} (s - x_2)_+^{d_1 - 1} \, \mathrm{d}s$ is the limit in $L^2(\mathbb{R}^2)$ of f_N (6.14): $\|f_N - f\| \to 0$ $(N \to \infty)$. See also Surgailis (1982) or Surgailis and Vaičiulis (1999) for details. This ends the proof of Theorem 6.1. □

Finally, we discuss the limit of partial sums' processes $S_{N1}(\tau) = \sum_{s=1}^{[N\tau]} X_s$. We shall suppose that conditions of Theorem 2.2 are satisfied and consider both cases when

$\{X_t\}$ exhibits either long or short memory in conditional mean. In other words, we shall assume that the coefficients g_j satisfy either condition

$$g_j \sim c_1 j^{d_1-1} \tag{6.16}$$

with some $0 < d_1 < \frac{1}{2}$, $c_1 \neq 0$, or condition

$$\sum_{j=0}^{\infty} |g_j| < \infty. \tag{6.17}$$

Conditions (6.16) and (6.17) imply

$$\operatorname{var} S_{N1} \sim \tilde{\kappa}_{11} N^{1+2d_1}, \tag{6.18}$$

$$\operatorname{var} S_{N1} \sim \tilde{\sigma}^2 N, \tag{6.19}$$

respectively, where $\tilde{\kappa}_{11} := \kappa_{11}/(d_1(1+2d_1))$, $\tilde{\sigma}^2 := \sum_{s\in\mathbb{Z}} \operatorname{cov}(X_s, X_0)$.

Theorem 6.2. *Assume conditions of Theorem* 2.2. *Moreover,*

(i) *if condition* (6.16) *holds* (*i.e.*, $\{X_t\}$ *exhibits long memory in conditional mean*), *then*

$$(\operatorname{var} S_{N1})^{-1/2} S_{N1}(\tau) \ \Rightarrow_{D[0,1]} \ J_1(\tau, d_1). \tag{6.20}$$

(ii) *if condition* (6.17) *holds* (*i.e.*, $\{X_t\}$ *exhibits short memory in conditional mean*), *then*

$$N^{-1/2} S_{N1}(\tau) \ \Rightarrow \ \tilde{\sigma} W(\tau), \tag{6.21}$$

where $\{W(\tau),\ \tau \geqslant 0\}$ *is a standard Brownian motion.*

(iii) *if condition* (6.17) *together with Assumption* A_4 *hold and there exist* $C < \infty$, $0 < d_2 < \frac{1}{2}$ *such that for all* $j \geqslant 1$

$$|h_j| \leqslant C j^{d_2-1}, \tag{6.22}$$

then convergence (6.21) *is true with* $\Rightarrow$ *replaced by* $\Rightarrow_{D[0,1]}$.

Proof. (i) (cf. Giraitis et al. (2000b, proof of Theorem 2.3).) We shall use the moving average representation (1.16). Write $Y_t = v'_{t,K} + v''_{t,K}$, where

$$v'_{t,K} := \zeta_t E\{A_t | \mathscr{F}_{[t-1,t-K]}\}, \quad v''_{t,K} := \zeta_t (A_t - E\{A_t | \mathscr{F}_{[t-1,t-K]}\})$$

and where $\mathscr{F}_{[s,t]} = \sigma\{\zeta_u\colon s \leqslant u \leqslant t\}$ is the σ-field. Then

$$X_t = \sum_{s\leqslant t} g_{t-s} Y_s = \sum_{s\leqslant t} g_{t-s} v'_{s,K} + \sum_{s\leqslant t} g_{t-s} v''_{s,K} =: X'_{t,K} + X''_{t,K}.$$

Note that both $\{v'_{t,K}\}$ and $\{v''_{t,K}\}$ are strictly stationary weak white noises, the former being finitely dependent; moreover, $E(v'_{t,K})^2 \to EY_0^2$, $E(v''_{t,K})^2 \to 0$ $(K \to \infty)$. Whence, one can easily show

$$N^{-(1/2+d_1)} \sum_{s=1}^{[N\tau]} X'_{s,K} \ \Rightarrow \ c_K J_1(\tau, d_1),$$

where $c_K^2 \to \tilde{\kappa}_{11}$ $(K \to \infty)$. On the other hand, $\operatorname{var}(N^{-1/2-d_1} \sum_{s=1}^N X''_{s,K}) \leqslant CE(v''_{0,K})^2 \to 0$ $(K \to \infty)$. Clearly, the above facts imply the convergence $(\operatorname{var} S_{N1})^{-1/2} S_{N1}(\tau) \Rightarrow$

$J_1(\tau, d_1)$. The tightness in $D[0,1]$ follows from stationarity of $\{X_t\}$ and (6.18) similarly as in the proof of Theorem 6.1.

(ii) follows by the same argument as (i), with $J_1(\tau, d_1), \tilde{\kappa}_{11}$ replaced by $W(\tau), \tilde{\sigma}^2$, respectively.

(iii) We need only to check the tightness, which follows from

$$ES_{N1}^4 \leqslant CN^{1+2d_2}. \tag{6.23}$$

Using Lemma 5.3 and (5.1), (6.17),

$$\begin{aligned}
ES_{N1}^4 &= E\left(\sum_{t=1}^{N}\sum_{s\leqslant t} g_{t-s}Y_s\right)^4 \\
&\leqslant C \sum_{1\leqslant t_1,\ldots,t_4\leqslant N}\ \sum_{s_1\geqslant s_2\geqslant s_3} |g_{t_1-s_1}g_{t_2-s_2}g_{t_3-s_3}g_{t_4-s_1}E[Y_{s_1}^2 Y_{s_2} Y_{s_3}]| \\
&\leqslant C \sum_{1\leqslant t_1,t_2,t_3\leqslant N}\ \sum_{s_1\geqslant s_2\geqslant s_3} |g_{t_1-s_1}g_{t_2-s_2}g_{t_3-s_3}|\,|s_1-s_2|_+^{d_2-1}|s_2-s_3|_+^{d_2-1} \\
&\leqslant C \sum_{u_1,u_2,u_3=0}^{\infty} |g_{u_1}g_{u_2}g_{u_3}| \sum_{1\leqslant t_1,t_2,t_3\leqslant N} |t_1-t_2+u_2-u_1|_+^{d_2-1}\,|t_2-t_3+u_3-u_2|_+^{d_2-1} \\
&\leqslant CN^{1+2d_2} \sum_{u_1,u_2,u_3=0}^{\infty} |g_{u_1}g_{u_2}g_{u_3}| \leqslant CN^{1+2d_2},
\end{aligned}$$

since $\sup_{v\in\mathbb{Z}} \sum_{\tau=1}^{N} |\tau+v|_+^{d_2-1} \leqslant CN^{d_2}$. This proves (6.23) and the theorem. □

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