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Lecture Notes

1 The Term Structure of Interest Rates: A Discrete Time Analysis

1.1 Fundamental Concepts

The *term* of a debt instrument with a fixed maturity date is the time until the maturity date. The **term structure** of interest rates at any time is the function relating interest rate to term. The study of the term structure inquires what market forces are responsible for the varying shapes of the term structure. In its purest form, this study considers only **bonds** for which we can disregard default risk[1], convertibility provisions[2], call provisions[3], floating rate provisions (provisions that change the interest payments according to some rule) or other special features. Thus, the study of the term structure may be regarded as the study *of the market price of time, over various intervals, itself.*

The term bond will be used here for any debt instrument, whether technically bond, bill, note, commercial paper, etc. and whether or not payments are defined in nominal (money) terms or in real terms (that is, tied to a commodity price index).

A bond represents a *claim* on a prespecified sequence of payments. A bond which is issued at time t and matures at time T is defined by a $w-element$ vector of payment dates $(t_1, t_2, ..., t_{w-1}, T)$, and by a $w-element$ vector of corresponding positive payments $(s_1, s_2, ..., s_w)$. In theoretical treatments of the term structure, payments may be assumed to be made continually in time, so that the payment stream is represented by a positive function of time $s(\tau)$, $t \leq \tau \leq T$.

Two kinds of payment sequences are common. For the **discount bond (or zero coupon bond)**[4]**,** the vector of payment dates contains a single element

[1] The default risk (or credit risk) on a bond is usually assessed in the form of a *credit rating*. There are two main services providing credit ratings: Moody's, and Standard and Poor's. UK government bonds have a negligible risk of default.

[2] That is there is the option (from either party) to exchange the bonds for shares.

[3] That is the issuer may decide to redeem the bond at an earlier date than maturity.

[4] *Zero coupon bonds* (or *zeros*, or *strips*) originated in 1982 from several investment bankers, and are among the most innovative instruments to appear in the credit market.

For example, Merrill Lynch issued in August 1982 the TIGRs (Treasury Investment Growth Receipts); this was followed within a matter of days by the CATS (Certificates of Accrual on Teasury Sacurities) program of Salomon Brothers. In January 1985 the Treasury announced that it would sponsor its own program called STRIPS (Separate Trading of Registered Interest and Principal of Securities). The STRIPS program subsequently displaced the private sector programs.

Simply stated, zeros divide the interest and principal obligations of a whole bond into individual, separated claims. Investors can then acquire the claims best suited to their needs.

T and the vector of payments contains the single element called the *principal (or face value).* A **coupon bond ,** in contrast, promises a payment at regular intervals of an amount c called the *coupon* and a payment of the last coupon and principal at the maturity date.

The purchaser at time t of a bond maturing at time T pays price $P(t,T)$ and is entitled to receive those payments corresponding to the t_i that are greater than t, so long as the purchaser continues to hold the bond. A coupon bond is said to be **selling at par** at time t if $P(t,T)$ is equal to the value of the principal, by our convention equal to 1.00 currency unit.

A coupon bond may be regarded as a portfolio of discount bonds. If coupons are paid once per time period, for example, then the portfolio consists of an amount c of discount bonds maturing at time $t+1$, an amount c of discount bonds maturing at time $t+2$, etc. and an amount $(c+1)$ of discount bonds maturing at time T. Should all such bonds be traded, we would expect, by the *law of one price*[5], that (disregarding discrepancies allowed by taxes, transactions costs and other market imperfections) the price of the portfolio of discount bonds should be the same as the price of the coupon bond.

There is thus (abstracting from market imperfections) a redundancy in bond prices, and if both discount and coupon bonds existed for all maturities, we could arbitrarily confine our attention to discount bonds only or coupon bonds only. (In practice, we do not always have prices on both kinds of bonds for the same maturities.) There is also a redundancy among coupon bonds, in that one can find coupon bonds of different coupon payments for the same maturity date.

At this stage, our analysis will be confined to default-free discount bonds. Coupon bearing bonds will be used in the analysis of duration, convexity, and fixed income portfolio management.

[5] That is two issues of identical risk and provisions should sell at the same price (this is a no arbitrage condition).

1.2 Basic Inputs.

- **The Spot Interest Rate,** prevailing at time t for repayment at time $(t+1) : r_t = R_t - 1$ or $(1 + r_t) = R_t$.

- The market price of a **Default-free Discount Bond** prevailing at time t for one currency unit at time T, denoted by $P(t, T)$. (So by definition $P(T, T) = 1$.)

- **The Yield-to-Maturity** viewed as the $(T - t)$ -period interest rate, compounded once per period, prevailing at time t, denoted by $Y(t, T)$. In other words, the yield is the internal rate of return on the bond (also called the "long rate").

- **The (one-period) Forward Rate**, that is, an interest rate embodied in current-time t prices that will prevail at time T for payment at time $(T+1)$, denoted by $f(t, T)$. So by definition we have that $Y(t, t+1) \equiv f(t, t)$. Also note that under certainty we have that $f(t, T) \equiv Y(T, T+1)$.

It is very helpful to interpret the first "time element" in bond quotations as the **evolutionary time,** and the second "time element" as the **maturity time.** Also, the **term (**remaining time to maturity) of the bond is *inversed time,* i.e. $(T - t)$.

1.3 Basic Relationships.

- Since the compound amount of principal $P(t,T)$ at interest rate $Y(t,T)$ after $(T-t)$ periods is 1 currency unit, it follows directly that:

$$\begin{aligned} P(t,T)(1+Y(t,T))^{T-t} &= 1, \text{ or} \\ P(t,T) &= \frac{1}{[1+Y(t,T)]^{T-t}}, \text{ or} \\ [1+Y(t,T)] &= \frac{1}{P(t,T)^{\frac{1}{T-t}}}. \end{aligned} \tag{1}$$

- Since,

$$[1+f(t,t)] \equiv [1+Y(t,t+1)] \overset{(1)}{=} \frac{1}{P(t,t+1)^{\frac{1}{t+1-t}}} = \frac{1}{P(t,t+1)} = 1+r_t,$$

  it follows that:

$$f(t,t) \equiv Y(t,t+1) \equiv r_t. \tag{2}$$

- Think of the following strategy: Being at the present time, t, buy (invest in

-) a $(T+1)$-period bond at $P(t,T+1)$, and sell (issue) an amount $\frac{P(t,T+1)}{P(t,T)}$ of T-maturity bonds at a price $P(t,T)$. Effectively, your net position at time t is zero, since your outlay of $P(t,T+1)$ is matched with your revenue, $\left[\frac{P(t,T+1)}{P(t,T)}\right] P(t,T)$. Implicitly, by this transaction you are committing yourself to invest at time T an amount $\frac{P(t,T+1)}{P(t,T)}$ of $(T+1)$-maturity bonds and then wait to receive your proceeds at time $(T+1)$. Then, the *forward rate,* or (under certainty) the *yield-to-maturity* of this contract is given by:

$$\begin{aligned} 1+f(t,T) &\equiv 1+Y(T,T+1) = \frac{1}{\left[\frac{P(t,T+1)}{P(t,T)}\right]}, \text{ or} \\ 1+f(t,T) &= \frac{P(t,T)}{P(t,T+1)}. \end{aligned} \tag{3}$$

  Observe that $f(t,T)$ is fully embodied in current prices!

- Equation (3) implies that:

$$\begin{aligned} (1+f(t,T-1)) &= \frac{P(t,T-1)}{P(t,T)} \Rightarrow \\ P(t,T) &= \frac{1}{1+f(t,T-1)} P(t,T-1). \end{aligned} \tag{4a}$$

Similarly,

$$\begin{aligned}(1+f(t,T-2)) &= \frac{P(t,T-2)}{P(t,T-1)} \Rightarrow \\ P(t,T-1) &= \frac{1}{1+f(t,T-2)}P(t,T-2). \end{aligned} \tag{4b}$$

Then, expressions (4a) and (4b) combined lead to:

$$P(t,T) = \frac{1}{[1+f(t,T-1)][1+f(t,T-2)]}P(t,T-2).$$

Recursively, we obtain that:

$$P(t,T) = \frac{1}{[1+r_t][1+f(t,t+1)]...[1+f(t,T-1)]}. \tag{5a}$$

Alternatively,

$$P(t,T) \overset{(1)}{=} \frac{1}{[1+Y(t,T)]^{T-t}} \overset{(5a)}{=} \frac{1}{[1+r_t]...[1+f(t,T-1)]}, \text{ or}$$

$$[1+Y(t,T)]^{T-t} = [1+r_t][1+f(t,t+1)]...[1+f(t,T-1)]. \tag{5b}$$

Note that yield and price are "global" concepts which can be expressed in terms of one-period forward rates.

1.4 No-Arbitrage Relationships and the Term Structure in a Certain Economy.

- (One-period investment strategy) Under no risk[6], any equilibrium must be characterized by

$$f(t,T) = r_T. \tag{6}$$

Otherwise, an arbitrage opportunity could exist through trading at time t in T, and $(T+1)-$maturity bonds.

Proof. Say $f(t,T) < r_T$. Then, buy a T-period bond at $P(t,T)$ and sell an amount $[1+f(t,T)]$ of $(T+1)$-period bonds at $P(t,T+1)$. Note that both actions take place at time t. The net cost, at time t, of this transaction is zero since (see eq. (3))

$$P(t,T) = P(t,T+1)[1+f(t,T)].$$

Note that the left-hand-side of the above expression is the cost of the investment whereas the right-hand-side is the revenue from the transaction. We have also used $[1+f(t,T)] = \frac{P(t,T)}{P(t,T+1)}$.

[6] This implies that future spot rates (r_{t+i}) are known with certainty, i.e. they are not random variables.

At time T the investor receives one currency unit which when reinvested at r_T will yield $(1 + r_T)$ at period $(T + 1)$.

The obligation of the investor is to repay at $(T + 1)$ the amount $[1 + f(t, T)]$. Since $f(t, T) < r_T$, an arbitrage profit is possible. ■

- (Multi-period investment strategy) Under no risk, any equilibrium must imply that "the certain **total** return of holding a T-period bond until it matures is equal to the total return on a series of one-period bonds over the term, $(T - t)$". From equation (5a) we have that:

$$P(t, T) = \frac{1}{(1 + r_t)[1 + f(t, t + 1)] \ldots [1 + f(t, T - 1)]},$$

which, after substituting eq. (6), can be written as follows:

$$P(t, T) = \frac{1}{(1 + r_t)(1 + r_{t+1}) \ldots (1 + r_{T-1})}. \tag{7a}$$

Equivalently,

$$\frac{1}{P(t, T)} \overset{(1)}{=} [1 + Y(t, T)]^{T-t} = (1 + r_t)(1 + r_{t+1}) \ldots (1 + r_{T-1}). \tag{7b}$$

Expression (7a) may also be recognized as the standard present value calculation with a changing but known interest rate.

- A direct consequence of equation (7b) is:

$$[1 + Y(t, T)] = [(1 + r_t)(1 + r_{t+1}) \ldots (1 + r_{T-1})]^{\frac{1}{T-t}}, \tag{8}$$

which means that "one plus the long-rate" is equal to the geometric mean of "one plus the short-rate".

- We finally need to evaluate the *total single period return* of a "long term" bond in a riskless economy. Using equation (7a) we can write that:

$$\frac{P(t + 1, T)}{P(t, T)} = [(1 + r_{t+1}) \ldots (1 + r_{T-1})]^{-1} [(1 + r_t) \ldots (1 + r_{T-1})] = (1 + r_t).$$

A single period return then must be equal to

$$\frac{P(t + 1, T)}{P(t, T)} = (1 + r_t) = R_t, \text{ or} \tag{9}$$

$$\frac{P(t + 1, T) - P(t, T)}{P(t, T)} = r_t. \tag{9'}$$

1.5 The Expectations Hypotheses in an Uncertain Economy.

1.5.1 The Unbiased Expectations Hypothesis (UEH).

It directly stems from the non-arbitrage condition (6) and takes the form:

$$E\left(\widetilde{r}_T\right) = f\left(t, T\right), \tag{10}$$

where $\widetilde{r}_T$ denotes the random (because of uncertainty) spot interest rate at time T.

Using eq. (5) and (10) we find that, in an economy characterized by the UEH, discount bond prices are given by:

$$P\left(t, T\right) = \frac{1}{\left(1 + r_t\right) E\left(1 + \widetilde{r}_{t+1}\right) ...E\left(1 + \widetilde{r}_{T-1}\right)}, \tag{11a}$$

or equivalently,

$$\left[\frac{1}{P\left(t, T\right)}\right]_{UEH} = \left[1 + Y\left(t, T\right)\right]^{T-t} = \left(1 + r_t\right) E\left(1 + \widetilde{r}_{t+1}\right) ...E\left(1 + \widetilde{r}_{T-1}\right). \tag{11b}$$

1.5.2 The Return-to-Maturity Expectations Hypothesis (RTM).

It directly stems from the non-arbitrage condition (7) and takes the following form:

$$\left[\frac{1}{P(t,T)}\right]_{RTM} = [1+Y(t,T)]^{T-t} = (1+r_t)\,E\left[\left(1+\tilde{r}_{t+1}\right)...\left(1+\tilde{r}_{T-1}\right)\right]. \tag{12}$$

Consider two traders, A and B, who trade according to their equilibrium beliefs: trader A uses the "mechanical" rule based on UEH, whereas trader B uses the more "economics" rule based on RTM. Under which condition(s) traders A and B will agree in their equilibrium prices? It follows directly from expressions (11b) and (12) that this will happen only when

$$E\left[(1+r_t)\left(1+\tilde{r}_{t+1}\right)...\left(1+\tilde{r}_{T-1}\right)\right] = (1+r_t)\,E\left(1+\tilde{r}_{t+1}\right)...E\left(1+\tilde{r}_{T-1}\right),$$

In other words, traders A and B will agree in their equilibrium prices when the future levels of short-term interest rates are *uncorrelated*.

However, empirical evidence suggests that short-term interest rates are *positively correlated* over time, in which case we have that:

$$E\left[(1+r_t)\left(1+\tilde{r}_{t+1}\right)...\left(1+\tilde{r}_{T-1}\right)\right] > (1+r_t)\,E\left(1+\tilde{r}_{t+1}\right)...E\left(1+\tilde{r}_{T-1}\right).$$

Therefore,

$$\begin{aligned} P_{RTM}(t,T) &< P_{UEH}(t,T)\,, \text{ and} \\ Y_{RTM}(t,T) &> Y_{UEH}(t,T)\,. \end{aligned}$$

Finance theorists tend to prefer the RTM to the UEH. The reason is that the RTM implies that, given a particular strategy by an investor, *total expected holding period returns* are equated by the RTM.

However**,** there is

an internal inconsistency in the RTM expectations theory. To illustrate it suppose that an investor buys at t an s-maturity bond $(t<s)$, and then reinvests the proceeds to a $(T-s)$- period bond $(s<T)$. The expected return of her portfolio over the entire holding period $(T-t)$ is:

$$\frac{1}{P(t,s)}E\left[\frac{1}{\tilde{P}(s,T)}\right] \overset{(12)}{=} E\left[(1+r_t)...\left(1+\tilde{r}_{s-1}\right)\right]E\left[\left(1+\tilde{r}_s\right)...\left(1+\tilde{r}_{T-1}\right)\right].$$

Alternatively, if she buys a $(T-t)$-period bond, she will earn

$$\frac{1}{P(t,T)} = E\left[(1+r_t)...\left(1+\tilde{r}_s\right)...\left(1+\tilde{r}_{T-1}\right)\right].$$

Again, the above two expressions will be equal as long as, for any s, the $(s-t)$-period and $(T-s)$-period yields are uncorrelated.

1.5.3 The Yield-to-Maturity Expectations Hypothesis (YTM).

It directly stems from the non-arbitrage condition (8) and takes the following form:

$$[1+Y(t,T)] = E\left\{\left[(1+r_t)\left(1+\tilde{r}_{t+1}\right)...\left(1+\tilde{r}_{T-1}\right)\right]^{\frac{1}{T-t}}\right\}, \tag{13a}$$

which implies that one-plus the long-rate equals the expected geometric mean of one-plus the short-rate. The above equation can be expressed as:

$$\left[\frac{1}{P(t,T)}\right]_{YTM} = [1+Y(t,T)]^{T-t} = \left[E\left\{\left[(1+r_t)...\left(1+\tilde{r}_{T-1}\right)\right]^{\frac{1}{T-t}}\right\}\right]^{T-t}, \tag{13b}$$

1.5.4 The Local Expectations Hypothesis (LEH).

It directly stems from the non-arbitrage condition (9) and takes the following form:

$$\frac{E[P(t+1,T)]}{P(t,T)} = (1+r_t), \tag{14}$$

i.e. the expected rate of return on any maturity bond over a single period is equal to the prevailing short rate. Cox, Ingersoll and Ross (1981, Econometrica) called this the **Risk-Neutral Expectations Hypothesis,** and showed that it is the only acceptable form in a competitive equilibrium.

Applying equation (14) recursively, we can obtain the following:

$$\begin{aligned} P(t,T) &= \frac{E[P(t+1,T)]}{1+r_t} = \frac{1}{(1+r_t)}E\left[\frac{P(t+2,T)}{\left(1+\tilde{r}_{t+1}\right)}\right] = ... \\ &= E\left[\frac{1}{(1+r_t)...\left(1+\tilde{r}_{T-1}\right)}\right], \end{aligned} \tag{15a}$$

or equivalently,

$$\left[\frac{1}{P(t,T)}\right]_{LEH} = \left\{E\left[\frac{1}{(1+r_t)...\left(1+\tilde{r}_{T-1}\right)}\right]\right\}^{-1}. \tag{15b}$$

One of the devastating implications of the Cox, Ingersoll and Ross (1981, Econometrica) critique was that in equilibrium only one of the forms of the expectations hypotheses should obtain since Jensen's inequality[7] implies that

[7] Jensen's inequality for *convex* functions implies that $E[F(X)] > F[E(X)]$.

the RTM, YTM, and LEH are mutually inconsistent. To see this consider that the LEH describes equilibrium. Then equation (15b) implies that:

$$\begin{aligned}
\left[\frac{1}{P(t,T)}\right]_{LEH} &= \left\{E\left[\frac{1}{(1+r_t)\ldots\left(1+\tilde{r}_{T-1}\right)}\right]\right\}^{-1} \\
&< E\left\{\left[\frac{1}{(1+r_t)\ldots\left(1+\tilde{r}_{T-1}\right)}\right]^{-1}\right\} = E\left[(1+r_t)\ldots\left(1+\tilde{r}_{T-1}\right)\right] \\
&\overset{(12)}{=} \left[\frac{1}{P(t,T)}\right]_{RTM},
\end{aligned}$$

i.e.

$$P(t,T)_{LEH} > P(t,T)_{RTM}.$$

Furthermore, equation (13b) implies that:

$$\begin{aligned}
P_{YTM}(t,T) &= \left[E\left\{\left[(1+r_t)\ldots\left(1+\tilde{r}_{T-1}\right)\right]^{\frac{1}{T-t}}\right\}\right]^{-(T-t)} \\
&< E\left[\frac{1}{(1+r_t)\ldots\left(1+\tilde{r}_{T-1}\right)}\right] = P_{LEH}(t,T),
\end{aligned}$$

i.e.

$$P(t,T)_{LEH} > P(t,T)_{YTM}.$$