Problem Set: BOPM.

1.Consider a one-period, two-state case in which XYZ stock is trading at $S_0 = \pounds 35$, has u of 1.05, and d of $1/1.05$, and for which the period risk-free rate is 2%.

a) Using the BOPM, determine the equilibrium price of an XYZ 35 European call option expiring at the end of the period.

b) Explain what an arbitrageur would do if the XYZ 35 European call was priced at £1.35 Show what the arbitrageur's cash flow would be at expiration when she closed.

c) Explain what an arbitrageur would do if the XYZ 35 European call was priced at £1.10. Show that the arbitrageur's cash flow would be at expiration when she closed.

a) The two payoffs in period one are:

$$C_u = \max(S_u - X, 0)$$
$$= 35 \times 1.05 - 35 = 1.75,$$
$$C_{ud} = \max(S_d - X, 0) = 0$$

The equilibrium price of the call is:

$$\begin{aligned} C_0 &= \frac{1}{r_f}[C_u p + C_d(1-p)] \\ &= \frac{1}{1.02}[1.75 \times 0.693] = 1.19, \end{aligned}$$

since $p = \frac{r_f - d}{u - d} = 0.693$

Alternatively, we have:

$$\begin{aligned} C_0 &= V_0 = H_0 S_0 - B_0 \\ &= 0.51 \times 35 - 16.74 = 1.19 \end{aligned}$$

since

$$\begin{aligned} H_u &= \frac{C_u - C_d}{S_0(u-d)} = \frac{1.75}{3.42} = 0.51, \\ B_u &= \frac{C_u d - C_d u}{r_f(u-d)} = \frac{1.67}{0.099} = 16.74. \end{aligned}$$

b) If the equilibrium price of the call in period zero is $C_0^m = 1.35 > C_0 = 1.19$

an investor will sell the call and receive: $C_0^m = 1.35$,

and (s)he will buy the portfolio and pay:

$V_0 = C_0 = 1.19$.

In period one (s)he will close his/her position

If we are in the up state then

(s)he will pay the payoff of the call: $C_u = 1.75$

and (s)he will receive the value of the portfolio:

$V_u = H_0 S_u - B_0 r_f = C_u = 1.75$.

Similarly, if we are in the down state

(s)he will pay the payoff of the call: $C_d = 0$

and (s)he will receive the value of the portfolio:

$$V_d = H_0 S_d - B_0 r_f = C_d = 0.$$

c) If the equilibrium price of the call in period zero is $C_0^m = 1.10 < C_0 = 1.19$

an investor will buy the call and pay $C_0^m = 1.10$,

and (s)he will sell the portfolio and receive:

$V_0 = C_0 = 1.19$.

In period one (s)he will close his/her position

If we are in the up state then

(s)he will receive the payoff of the call: $C_u = 1.75$

and (s)he will pay the value of the portfolio:

$V_u = H_0 S_u - B_0 r_f = C_u = 1.75$.

Similarly, if we are in the down state

(s)he will receive the payoff of the call: $C_d = 0$

and (s)he will pay the value of the portfolio:

$V_d = H_0 S_d - B_0 r_f = C_d = 0.$

2. Assume two periods to expiration, $u = 1.05$, $d = 1/1.05$, $r_f = 1.02$, $S_0 = \pounds 50$, no dividends, and $X = \pounds 50$ on a European call expiring at the end of the second period.

a) Find: C_{uu}, C_{ud}, C_{dd}, C_d, C_u, C_0, H_u, H_d, H_0, B_u, B_d, and B_0.

b) Overpriced case:

i) Define the arbitrage strategy you would employ if the current market price of the call was £2.60. Assume any positive cash flow is invested in a risk-free security.

ii) What would be your cash flow from your arbitrage in period 1 if you closed your initial strategy when the stock was priced at S_u and the call was selling at $C_1^m = \pounds 3.75$?

How would you readjust your initial strategy to avoid a loss? Show what your cash flows would be when you closed at the end of the second period (assume the stock will follow the binomial process and will be either uuS_0 or udS_0).

c) Underpriced case:

i) Define the arbitrage strategy you would employ if the current market price of the call was £2.20. Assume any positive cash flow is invested in a risk-free security.

ii) What would be your cash flow from your arbitrage investment in period 1 if you closed your initial position when the stock was priced at S_u and the call was selling at $C_1^m = £3.25$?

How would you readjust your initial strategy to avoid the loss? Show what your cash flow would be when you closed at the end of the second period (assume the stock will follow the binomial process and will be either uuS_0 or udS_0).

a) First we calculate the three call payoffs in period two:

$$\text{2}$$

$$C_{uu} = \max(\overbrace{S_{uu}}^{u^2 S_0} - X, 0)$$
$$= 5.125$$
$$C_{ud} = \max(\overbrace{S_{ud}}^{udS_0} - X, 0) = 0$$
$$C_{dd} = \max(\overbrace{S_{dd}}^{ddS_0} - X, 0) = 0$$

and then, using the one period binomial model, the two payoffs in period 1 are:

$$\text{1}$$

$$C_u = \frac{1}{r_f}[C_{uu}p + C_{ud}(1-p)]$$
$$= \frac{1}{1.02}[5.125 \times 0.693 + 0] = 3.480$$
$$C_d = \frac{1}{r_f}[C_{ud}p + C_{dd}(1-p)] = 0$$

where we used $p = \frac{r_f - d}{u - d} = 0.693$.

Next, using again the one binomial model, we calculate the equilibrium price of the call in period 0:

$$\begin{aligned} C_0 &= \frac{1}{r_f}[C_u p + C_d(1-p)] \\ &= \frac{1}{1.02}[3.480 \times 0.693 + 0] = 2.363. \end{aligned}$$

Alternatively, one can use the binomial formula for the two period model:

$$\begin{aligned} C_0 &= \frac{1}{r_f^2}[C_{uu}p^2 + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}] \\ &= \frac{1}{(1.02)^2}[5.125 \times (0.693)^2] = 2.363 \end{aligned}$$

The H_u, H_d, and H_0 are given by

$$
\begin{aligned}
H_u &= \frac{C_{uu} - C_{ud}}{S_0(u^2 - ud)} = \frac{5.125}{50 \times 0.103} = 1, \\
H_d &= \frac{C_{ud} - C_{dd}}{S_0(u^2 - ud)} = 0, \\
H_u &= \frac{C_u - C_d}{S_0(u - d)} = \frac{3.480}{4.881} = 0.713
\end{aligned}
$$

The B_u, B_d, and B_0 are given by

$$
\begin{aligned}
B_u &= \frac{C_{uu}d - C_{ud}u}{r_f(u - d)} = \frac{4.881}{0.099} = 49.02, \\
B_d &= \frac{C_{ud}d - C_{dd}u}{r_f(u - d)} = 0, \\
B_u &= \frac{C_u d - C_d u}{r_f(u - d)} = \frac{3.315}{0.099} = 33.289.
\end{aligned}
$$

b) i) If the price of the call in period zero is $C_0^m = 2.60$, that is greater than its equilibrium value, 2.36,

then in period 0 an investor will sell the call

and receive $C_0^m = 2.60$

and (s)he will buy the portfolio and pay

$V_0 = H_0 S_0 - B_0 = C_0 = 2.36$

The overall position in period one will be:

$C_0^m - C_0 = 2.60 - 2.34 = 0.26.$

ii) If in period one $C_1 = 3.75 > C_u = 3.48$, assuming that we are in the up state,

and the investor will close his/her position,

that is, (s)he will buy the call and pay $C_1 = 3.75$

and (s)he will sell the portfolio and receive:

$V_u = H_0 S_u - B_0 r_f = C_u = 3.48$

then his/her overall position will be

$V_u - C_1 = 3.48 - 3.75 = 0.27$

In period 1 instead of closing the position,

(s)he has to buy: $H_u - H_0$ stocks

and thus borrow an amount of money $(H_u - H_0)S_u$

the overall borrowing will be:

$$
\begin{aligned}
(H_u - H_0)S_u + B_0 r_f &= H_u S_u - \overbrace{(H_0 S_u - B_0 r_f)}^{V_u = H_u S_u - B_u} \\
H_u S_u - (H_u S_u - B_u) &= B_u = 49.020
\end{aligned}
$$

In period two the investor will close her/his position:

If we are in the up state then:

(s)he will pay the payoff of the call: $C_{uu} = 5.125$

and (s)he will receive the value of the portfolio:

$V_{uu} = H_u S_{uu} - B_u r_f = 5.125$

So his/her overall position is $V_{uu} - C_{uu} = 0$

Similarly, if we are in the down state then:

(s)he will pay the payoff of the call: $C_{ud} = 0$

and (s)he will receive the value of the portfolio:

$V_{ud} = H_u S_{ud} - B_u r_f = 0$

So his/her overall position is $V_{ud} - C_{ud} = 0$

c) i) If the price of the call in period zero is $C_0^m = 2.20$, that is less than its equilibrium value, 2.36,

then in period 0 an investor will buy the call

and pay $C_0^m = 2.20$

and (s)he will sell the portfolio and receive

$V_0 = H_0 S_0 - B_0 = C_0 = 2.36$

The overall position in period one will be:

$C_0 - C_0^m = 2.36 - 2.20 = 0.16.$

ii) If in period one $C_1 = 3.25 < C_u = 3.48$, assuming that we are in the up state,

and the investor will close his/her position,

that is, (s)he will sell the call and receive $C_1 = 3.25$

and (s)he will buy the portfolio and pay:

$V_u = H_0 S_u - B_0 r_f = C_u = 3.48$

then his/her overall position will be

$C_1 - V_u = 3.25 - 3.48 = -0.23$

In period 1 instead of closing the position,

(s)he has to sell: $H_u - H_0$ stocks

and thus receive an amount of money $(H_u - H_0)S_u$

the overall amount will be:

$$
\begin{aligned}
(H_u - H_0)S_u + B_0 r_f &= H_u S_u - \overbrace{(H_0 S_u - B_0 r_f)}^{V_u = H_u S_u - B_u} \\
H_u S_u - (H_u S_u - B_u) &= B_u = 49.020
\end{aligned}
$$

In period two the investor will close her/his position:

If we are in the up state then:

(s)he will receive the payoff of the call: $C_{uu} = 5.125$

and (s)he will pay the value of the portfolio:

$V_{uu} = H_u S_{uu} - B_u r_f = 5.125$

So his/her overall position is $C_{uu} - V_{uu} = 0$

Similarly, if we are in the down state then:

(s)he will receive the payoff of the call: $C_{ud} = 0$

and (s)he will pay the value of the portfolio:

$V_{ud} = H_u S_{ud} - B_u r_f = 0$

So his/her overall position is $C_{ud} - V_{ud} = 0$

3. Suppose XYZ stock is trading at $S_0 = \pounds 101$, $u = 1.02$, $d = 1/1.02$, the period risk-free rate is 1%, and the stock pays no dividends. Using the n-period BOPM, determine the equilibrium price of an XYZ 100 European call expiring at the end of the third period ($n = 3$).

In the n-period BOPM the equilibrium price of the call is

$$\begin{aligned} C_0 &= \frac{1}{r_f^n}\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}C_{u^j d^{(n-j)}} \\ &= \frac{1}{r_f^n}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}p^j(1-p)^{n-j} \\ &\quad \max[S_0 u^j d^{(n-j)} - X, 0] \end{aligned}$$

If $n = 3$ (we have a 3 period Binomial model) the form will give us

$$\begin{aligned} C_0 &= \frac{1}{r_f^3}[p^3 C_{uuu} + 3p^2(1-p)C_{uud} \\ &\quad +3p(1-p)^2 C_{udd} + (1-p)^3 C_{ddd}] \\ &= \ldots \end{aligned}$$

since: $\binom{3}{0} = \frac{3!}{0!3!} = 1$, $\binom{3}{1} = \frac{3!}{2!1!} = 3$, $\binom{3}{2} = \frac{3!}{2!1!} = 3$, $\binom{3}{3} = \frac{3!}{3!0!} = 1$.