Brunel University
Msc., EC5504, Financial Engineering
Prof Menelaos Karanasos

Lecture Notes: The Binomial Pricing Model

1 A One-Step Binomial Model

Consider a stock whose price is initially S_0. We are interested in deriving the current price (f_0) of a European call option on the stock. Suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level, S_0u, or down from S_0 to a new level, S_0d (so $u > 1$, $d < 1$). Thus the movement of the stock price can be described by the so called *one-step binomial tree.* At time T, let the payoff from the option be f_u if the stock price moves up, and f_d if the stock price moves down. In what follows we price the option by assuming that no arbitrage opportunities exist.

First, we set up a portfolio of the stock and the option in such a way that there is no uncertainty about the value of the portfolio at expiration of the option, i.e. at time T. In particular, we consider a portfolio consisting of a long position in Δ shares and a short position in one option. We calculate the value of Δ that makes the portfolio riskless:

$$\underbrace{S_0u\Delta - f_u}_{\substack{\text{value of portfolio} \\ \text{when } S \text{ moves up}}} = \underbrace{S_0d\Delta - f_d}_{\substack{\text{value of portfolio} \\ \text{when } S \text{ moves down}}} \Rightarrow \Delta = \frac{f_u - f_d}{S_0(u-d)}. \tag{1}$$

The above equation shows that Δ is the ratio of the change in the option price to the change in the stock price as we move between nodes.

The absence of arbitrage opportunities implies that the fair price of any investment is given by the present value of its future payoff. So we should have that the cost of setting up the above portfolio is equal to the present value of its future value. Since the portfolio has no risk, we should use the risk-free interest rate (r) to discount any future payments. In other words, a riskless portfolio must earn the risk-free interest rate. Therefore, we have

$$\underbrace{S_0\Delta - f_0}_{\substack{\text{cost of the} \\ \text{portfolio}}} = \underbrace{(S_0u\Delta - f_u)\, e^{-rT}}_{\substack{\text{present value of} \\ \text{future payoff}}} \tag{2}$$

or

$$f_0 = S_0\Delta - (S_0 u\Delta - f_u)\, e^{-rT}. \tag{2'}$$

Substitute (1) into the above to get

$$f_0 = e^{-rT} \underbrace{[p f_u + (1-p)\, f_d]}_{\text{expected future value}}, \tag{3}$$

where

$$p = \frac{e^{rT} - d}{u - d}. \tag{4}$$

It is natural to interpret the variable p given by eq. (4) as the probability of an up movement in the stock price, and the variable $1-p$ as the probability of a down movement in the stock price. Thus, eq. (3) states that the value of the option today is its expected future value discounted at the risk-free interest rate.

The expected stock price at time T is

$$\begin{aligned}
E(S_T) &= pS_0u + (1-p)\, S_0 d \\
&= pS_0 (u - d) + S_0 d \overset{\text{using (4)}}{\Longrightarrow} \\
E(S_T) &= \left(\frac{e^{rT} - d}{u - d}\right) S_0 (u - d) + S_0 d \Rightarrow \\
E(S_T) &= S_0 e^{rT}. && (5)
\end{aligned}$$

the above shows that the stock price grows, on average, at the risk-free rate. Therefore, setting the probability of an up movement equal to p, is equivalent to assuming that the per annum rate of return on the stock equals the risk-free rate.

In a risk neutral world all individuals are indifferent to risk. They require no compensation for risk, and the expected return on all securities is the risk-free rate. Equation (5) demonstrates that we are assuming a risk-neutral world when we set the probability of an up movement to p. This result is an example of an important general principle in option pricing known as **risk neutral valuation.** The principle shows that it is valid to assume the world is risk neutral when pricing options. However, the resulting option prices are correct not just in a risk neutral world, but in the real world as well.

2 A Two-Step Binomial Model

The above analysis can be extended when the movement of the stock price can be described by a *two-step binomial tree.* The stock price is initially S_0.

During each time step it either moves up to u times the previous price or moves down to d times the previous price. The option on the stock matures at T and the length of each time step is $T/2$. The value of the option after, say, two down movements is denoted by f_{dd} (see Figure 1).

After repeated application of eq. (3) we obtain:

$$
\begin{aligned}
f_0 &= e^{-r\frac{T}{2}} \left[p f_u + (1-p) f_d\right], \text{ where} &\quad (6a)\\
f_u &= e^{-r\frac{T}{2}} \left[p f_{uu} + (1-p) f_{ud}\right], \text{ and} &\quad (6b)\\
f_d &= e^{-r\frac{T}{2}} \left[p f_{ud} + (1-p) f_{dd}\right]. &\quad (6c)
\end{aligned}
$$

Substitution of (6b) and (6c) into (6a) gives

$$f_0 = e^{-rT} \left[p^2 f_{uu} + 2p(1-p) f_{ud} + (1-p)^2 f_{dd}\right], \quad (7)$$

where

$$\left\{\begin{array}{c} p^2 \\ 2p(1-p) \\ (1-p)^2 \end{array}\right\} \text{ is the probability of } \left\{\begin{array}{c} \text{two up} \\ \text{one up and one down} \\ \text{two down} \end{array}\right\} \text{ movements of } S.$$

Note that eq. (7) states that the option price is equal to its expected payoff discounted at the risk-free interest rate. This is consistent with the principle of risk neutral valuation which continues to hold as we add more steps to the binomial tree.

The above procedure can be used to price any derivative dependent on a stock whose price changes are binomial.

► Example 1 Consider a two-year European put option, i.e. $T = 2$, with $X = 52$, on a stock with $S_0 = 50$. Suppose that there are two time steps of equal size, and in each time step the stock price moves either up or down by 20%. The continuously compounded risk-free rate is 5% per annum.

► From eq. (4) we get $p = 0.6282$. Note that the possible final stock prices are 72, 48, and 32. So $f_{uu} = 0$, $f_{ud} = 4$, and $f_{dd} = 20$. Eq. (7) gives the option premium: $f_0 = 4.1923$, whereas eq. (6b)-(6c) give $f_u = 1.4147$ and $f_d = 9.4636$. (See Figure 2.)

2.1 American Options

In order to value an American option, we need to work back through the binomial tree from the end to the beginning, testing at each node to see whether early exercise is optimal. The value of the option at the final nodes is the same as for the European option. At earlier nodes the value of the option is the greater of (i) the value of the European option, and (ii) the payoff from early exercise. In the context the two-step binomial tree, we have that

$$\begin{aligned} f_u &= \max\left(e^{-r\frac{T}{2}}\left[pf_{uu} + (1-p)\,f_{ud}\right],\ S_0u - X\right), \\ f_d &= \max\left(e^{-r\frac{T}{2}}\left[pf_{ud} + (1-p)\,f_{dd}\right],\ S_0d - X\right). \end{aligned}$$

► Example Suppose that the example given above refers to an American put. The stock prices and their probabilities remain unchanged. Also the values for the option at the final nodes remain the same as those for the European put. Furthermore, f_u remains unchanged, whereas $f_d = 12$ (this is because it is optimal to exercise at this node). Finally $f_0 = 5.0894$. (See Figure 3.)

The above binomial models are very simple. An analyst can expect to obtain a very rough approximation to an option price by assuming that stock price movements during the life of the option consist of one or two binomial steps. When binomial trees are used in practice, the life of the option is typically divided into 30 or more time steps of equal length. In each time step there is a binomial stock price movement. The parameters u and d are chosen to match the stock price volatility.