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Lecture Notes: The "Greeks"

A common misconception about option pricing models is that their typical use by traders is to indicate the "right" price of an option. Option pricing models do not necessarily give the "right" price of an option. The "right" price of an option is what someone is willing to pay for a particular option. So far the best pricing method is the market; an efficient market will give the best prices for options. The true benefits of an options pricing model are twofold: (i) it provides an accurate "snap shot" of current market conditions, and (ii) it breaks the option's market price into each of the factors that comprise it. Therefore, an options pricing model can be used to predict how the market price of the option should change given that any of the factors that determine the value of an option change (i.e. the underlying instrument market price and volatility, the time to expiration, and the interest rates). Below we discuss what are commonly referred to as the "Greek letters", or simply the "Greeks". The "Greeks" measure the option's price sensitivity to changes in the factors that determine the value of an option.

Recall that the Black and Scholes stock option pricing model is given by

$$\text{European call premium} \; : \; C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2), \tag{1}$$

$$\text{European put premium} \; : \; P_0 = -S_0 N(-d_1) + Xe^{-rT} N(-d_2), \tag{2}$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \text{ and} \tag{3a}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}, \tag{3b}$$

where T is time to expiration, expressed as a proportion of the year; r is the continuously compounded risk-free rate of return, expressed annually (if x is the simple annual return on a riskless security, such as a Treasury bill with maturity equal to the call's expiration date, then the continuous compounded rate is $\ln(1+x)$); σ is the annualized standard deviation of the return (the return is computed using the log of price relatives $\ln(S_t/S_{t-1})$); $N(d_1)$, $N(d_2)$ are cumulative normal probabilities.

Consider a securities firm which trades its own money and is concerned with managing the risk of its position. Using a pricing model, such as the BS one, the firm might identify options that are mispriced and have its traders to sell (buy) the overpriced (underpriced) options. The executive trader is often asked to make a first pass at hedging each trade by immediately effecting an opposite position in a fairly priced option or in the underlying security, so that the position composed of the initial and hedge trades is insensitive to a small price change in the underlying security. This type of hedge has been labeled with the Greek letter *delta* (Δ) and has come to be known as a *delta neutral hedge.*

The firm's risk management group job is to step beyond the first hedge and make the firm's position insensitive to larger price changes in the underlying security and to changes in all of the other pricing model's parameters. These other sensitivities have also been given Greek letter names: *gamma* (Γ) for the sensitivity of Δ to a small price change in the underlying security; *theta* (Θ) for the sensitivity of the option's price to time to expiration; *kappa* (κ), or sometimes *vega,* for the sensitivity of the option's price to a small change in the volatility of the underlying asset; and *rho* (ρ) for the sensitivity of the option's price to a small change in the short-term riskless interest rate.

Note that in what follows we give the equations for the "Greeks" of European options on non-dividend-paying stocks.

▶ **Black and Scholes Hedged Portfolio for Calls.**

In the BS model we can construct a hedged portfolio by finding the proportion of shares of stock per written call that makes the value of the portfolio invariant to small changes in the stock price. The value of the hedged portfolio (V_h) is

$$\underbrace{V_h}_{\substack{\text{value of}\\\text{portfolio}}} = \underbrace{n_c C}_{\substack{\text{call}\\\text{position}}} + \underbrace{n_s S}_{\substack{\text{stock}\\\text{position}}},$$

where n_c is the number of shares, and n_s is the number of calls. To find the hedged portfolio we first need constrain the portfolio to consist of one written call, so that

$$n_c = -1,$$

and then find the value of n_s which makes the partial derivative of V_h with respect to S equal to zero:

$$\frac{\partial V_h}{\partial S} = n_c \frac{\partial C}{\partial S} + n_s \frac{\partial S}{\partial S} = 0.$$

So we get

$$(-1)\frac{\partial C}{\partial S} + n_s = 0 \Rightarrow n_s = \frac{\partial C}{\partial S} = N(d_1).$$

Thus in the BS model a hedged portfolio can be formed by selling a call and buying $N(d_1)$ shares of stock or a multiple of this strategy.

Example: Consider a call with $S = \$45$, $X = \$50$, $T = 0.25$, $\sigma = 0.5$, $r = 0.06$. Using the BS formula we compute $N(d_1) = 0.4066$, and $C = \$2.88$. To form a hedged portfolio would require selling one call and buying 0.4066 shares. If we assume that the market price equals the BS price, then a \$1 increase (decrease) in the price of the stock would lead to a \$0.4066 increase (decrease) in the value of the stock position. However, this increase (decrease) would be offset by a \$0.4066 decrease (increase) in the short call position.

- **The Delta (Directional Risk):**

$$\Delta = \frac{\partial C}{\partial S} = N(d_1), \text{ for a European Call}$$
$$\Delta = \frac{\partial P}{\partial S} = N(d_1) - 1, \text{ for a European Put}$$

  Delta is used to measure an option's price sensitivity to a small change in the price of the underlying stock. Note that the delta is also known as a *hedge ratio* (see the discussion above). Using delta hedging for a short position in a European call option involves keeping a long position of $N(d_1)$ shares at any given time. On the other hand, a short position in a put option should be hedged with a short position in the underlying stock (since $N(d_1) - 1$ is negative). Figure 1 plots typical patterns of option deltas.

▶ The delta measures the slope of the curved relationship between the option's price and the price of the underlying market.[1]

▶The delta for a call option is positive, ranging in value from approximately zero for deep out-of-the money calls to approximately 1 for deep in-the-money calls. In contrast, the delta for a put option is negative, ranging from approximately 0 to -1. The delta of an option which is at-the-money is always approximately 0.5.

▶ Deltas change in response not only to stock price changes, but also with respect to time to expiration. As the time to expiration decreases the delta of an in-the-money option increases, while an out-of-the money call or put tends to decrease.

▶ The option's delta can also be used to measure the probability that the option will be in-the-money at expiration**.** Thus, a call with a $\Delta = N(d_1) = 0.40$ has approximately 40% chance that its stock price will exceed the option's exercise price at expiration.

[1] Hence, at expiration the option will have either a slope of zero (if out-of-the-money) or one (if in-the-money).

The delta shows how the option's price changes when the underlying price changes. So we can think of delta as the "speed" of the option. Speed is generally given by the first derivative. Another important performance feature is acceleration. Acceleration is generally measured by the second derivative. Thus the "acceleration" of the option can be found by taking the derivative of the option's delta with respect to the underlying price. This is commonly referred to as the "gamma".

- **The Gamma (Curvature Risk):**

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}, \text{ for a European Call;}$$
$$\Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}, \text{ for a European Put;}$$

where $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\left(x^2/2\right)}$ (i.e. $N'(d_1)$ gives the height of the standard normal density function at the horizontal coordinate d_1). *Gamma measures the change in an option's delta for a small change in the stock price, i.e. gamma is the second derivative of the option's price with respect to the stock price.* Gamma can be thought of as a measure of the exposure the position has to a change in the actual volatility of the underlying market. Figure 2 plots typical patterns of option gammas.

► Intuitively, gamma jointly measures how close the current market is to the strike price of the option and how close the option is to expiration. The closer the market price is to the strike price and the closer the maturity of the option is to the expiration date, the higher gamma will be.

► For an at-the-money option, gamma increases as the time to maturity decreases. Short-life at-the-money options have very high gammas, meaning that the value of the options holder's position is highly sensitive to jumps in the stock price.

► All buyers of options gain from movements in the price of the underlying asset. For this reason, holders of options are often referred to as being "long gamma". For sellers of options, the gamma exposure is exactly opposite to that of buyers of options. Those who sell options can be hurt when the gamma is high and the underlying market price moves. Therefore, they are often referred to as being "short gamma".

► An underlying contract has a delta of 1.00. What is the gamma associated with an underlying contract? Since the gamma is the rate of change in the delta, and the delta of an underlying contract is always 1.00, the gamma must be zero.

- **The Theta:**

$$\Theta = -\frac{\partial C}{\partial T} = -\frac{S_0 N'(d_1)\,\sigma}{2\sqrt{T}} - rXe^{-rT}N(d_2)\,, \text{ for a European Call;}$$
$$\Theta = -\frac{\partial P}{\partial T} = -\frac{S_0 N'(d_1)\,\sigma}{2\sqrt{T}} + rXe^{-rT}N(-d_2)\,, \text{ for a European Put;}$$

where $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\left(x^2/2\right)}$ (i.e. $N'(d_1)$ gives the height of the standard normal density function at the horizontal coordinate d_1). *Theta is the change in the price of an option with respect to the passage of time, with all other factors constant. Equivalently, it is the rate of change of the option price with respect to a decrease in the time to maturity. Theta is also referred to as the time decay of the option.* We can thus think of theta as a measure of the exposure the position has to the passage of time. Figure 3 plots typical patterns of option thetas.

▶ The theta is usually negative for an option. This is because as time passes, the option tends to become less valuable.

▶ Theta is the slope of the time decay curve (see LN3) at a particular point.

▶ Like delta, theta changes in response to changes in the stock price and time to expiration. When the stock price is very low theta is close to zero. For an at-the-money call option, theta is large and negative. As the stock price increases, theta tends to rXe^{-rT}.

▶ Theta is not the same type of hedge parameter as delta. There is uncertainty about the future stock price, but there is no uncertainty about the passage of time. So it does not make any sense to hedge against the effect of the passage of time on an option. Below we will see why traders regard theta as a useful descriptive statistic for a portfolio.

Example: Suppose that $S_0 = 45$, $S = 50$, $T = 0.25$, $\sigma = 0.5$, $r = 0.06$; so $N(d_1) = N(-0.2364) = 0.4066$, $N(d_2) = N(0.4846) = 0.3131$, and $N'(d_1) = 0.3879$. In this case, the call has a theta of -7.8024, and the put has a theta of -6.698. This means that if the option is held 1% of a year (approximately 2.5 trading days) and there is no change in the stock's price, then the call would decline in value by approximately 0.078 currency units and the put would decrease by 0.06698 currency units.

Volatilities change over time. This means that the value of the derivative is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time.

- **The Kappa (Vega):**[2]

$$
\begin{aligned}
\kappa &= \frac{\partial C}{\partial \sigma} = S_0\sqrt{T}N'(d_1)\,, \text{ for a European Call;} \\
\kappa &= \frac{\partial P}{\partial \sigma} = S_0\sqrt{T}N'(d_1)\,, \text{ for a European Put;}
\end{aligned}
$$

  where $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\left(x^2/2\right)}$ (i.e. $N'(d_1)$ gives the height of the standard normal density function at the horizontal coordinate d_1). *Kappa is the change in the price of an option with respect to a small change in the volatility of the underlying asset, with all other factors constant.* So kappa is a measure of the exposure the position has to a change in the implied volatility of the options market. In other words, kappa represents the amount of money you make or lose from a given change in volatility. Figure 4 plots typical patterns of option kappas (vegas).

▶ Whenever we purchase options we are purchasing the time value component as well as the intrinsic value component of the premium. When the implied volatility rises, the time value and the total option premium also rise. Since options gain value with rising volatility, the kappa of a long position in a call or put option is always positive. By symmetry, when one sells options, they benefit from a decrease in the implied volatility and therefore have a kappa negative exposure.

▶ At-the-money options have a greater kappa than either in-the-money or out-of-the-money options with the same time to expiration.

▶ Whereas the gamma of an at-the-money option increases as the time to maturity date approaches, the reverse is true for kappa. Therefore, a longer term option will always be more sensitive to a change in volatility than a short-term option.

Example: Consider an option with $\frac{\partial C}{\partial \sigma} = 66.44$. Thus a 1% increase in volatility (from 25% to 26%) increases the value of the option by approximately 0.6644 (=0.01×66.44). Usually kappa is quoted as the change in value per 1% change in volatility. So in this case $\kappa = 0.6644$.

[2] This derivative is also called zeta, sigma, omega, epsilon and a variety of other Greek letters. Although vega is not a Greek letter, it is quite often preferred by authors because it has a "V" for volatility and sounds like Greek.

Example: If an option has $\kappa = 0.15$, for each percentage point increase (decrease) in volatility, the option will gain (lose) 0.15 in theoretical value. If the option has a theoretical value of 3.25 at a volatility of 20%, then it will have a theoretical value of 3.40 at a volatility of 21%, and a theoretical value of 3.10 at a volatility of 19%.

- **The Rho:**

$$\begin{aligned} \rho &= \frac{\partial C}{\partial r} = XTe^{-rT} N(d_2) > 0, \text{ for a European Call;} \\ \rho &= \frac{\partial P}{\partial r} = -XTe^{-rT} N(-d_2) < 0, \text{ for a European Put.} \end{aligned}$$

  Rho captures the sensitivity of the option's theoretical value to changes in short-term interest rates. Thus rho is a measure of the exposure the position has to a change in interest rates from today until expiration. Figure 5 plots typical patterns of option rhos.

▶ Consider a costly underlying asset, and recall that buying a call option gives you the right to hold the underlying asset and costs less than actually investing in the underlying market. Thus the remaining balance can be placed on deposit to earn interest. In other words the call option increases the leverage and frees up cash to place on deposit. As short-term interest rates rise, this leverage factor becomes more and more attractive. To remain at equilibrium, the call option's price must increase when the short-term interest rate rises to make the investor indifferent to either investing in the option/deposit trade or the underlying market.[3]

▶ Typically, options that are deeply in-the-money will have the greatest rho since they require the greatest cash outlay.

▶ The greater the amount of time to expiration, the greater the rho.

Example: Consider a put option with rho equal to −42.57. This means that for a one percentage point increase in the risk-free interest rate (from 8% to 9%) the value of the option decreases by 0.4257 (=0.01×42.57).

[3] If the underlying asset is "costless" (e.g. forward contract in foreign exchange), then as interest rates rise the values of both call and put options fall.

Now consider the case where the underlying asset is "neutral". By that we mean that both the underlying asset and the options are "costless" (e.g. options on futures; these do not require payment when purchased but are margined like the underlying futures), then interest rates have no impact on options prices.

The description of call and put options in terms of their "Greek" values can be extended to option positions. For example, consider an investor who holds n_1 calls at a price of C_1 per call and n_2 calls on another option on the same stock at a price of C_2 per call. The value of this portfolio (V_P) is:

$$V_P = n_1 C_1 + n_2 C_2. \tag{4}$$

- The **Position Delta** is obtained by differentiating (4) with respect to S :

$$\begin{aligned} \Delta_P &= \frac{\partial V_P}{\partial S} = n_1 \left(\frac{\partial C_1}{\partial S} \right) + n_2 \left(\frac{\partial C_2}{\partial S} \right), \text{ or} \\ \Delta_P &= n_1 \Delta_1 + n_2 \Delta_2. \end{aligned} \tag{5}$$

  The position delta measures the change in the portfolio's value in response to a small change in the stock, with other factors being constant. A **neutral delta position** can be constructed with a value invariant to small changes in the stock price, i.e. set Δ_P equal to zero:

$$\Delta_P = 0 \Rightarrow n_1 = -\left(\frac{\Delta_2}{\Delta_1} \right) n_2. \tag{6}$$

► The above popular neutral delta strategy is also known as the *neutral ratio spread*. This spread position is formed with the proportion of short calls(or puts) to long being such that the portfolio is invariant to stock price changes (position delta of zero). For example, if the hedge ratio for a July 50 call is $N(d_1)_1 = 0.570$ and the hedge ratio for a July 60 call on the same stock is $N(d_1)_2 = 0.292$, then the *ratio spread* (also referred to as a *delta spread*) needed to form a riskless position is (using equation (6)):

$$-\frac{N(d_1)_2}{N(d_1)_1} = -\frac{0.292}{0.570} = -0.5123.$$

A *riskless ratio spread* is used to define arbitrage or quasi-arbitrage strategies, for example, a neutral ratio spread formed by buying an underpriced call and selling an overpriced or equal-in-value call on the same stock. Given that $N(d_1)$ changes as the stock price and time change, this ratio spread strategy would need to be readjusted each period to keep the spread riskless until it is profitable to close.

- The **Position Gamma** (Γ_P) defines the change in the position's delta for a small change in the stock price, with other factors being held constant. It is found by differentiating (6) with respect to S :

$$\begin{aligned} \Gamma_P &= \frac{\partial \Delta_P}{\partial S} = n_1 \left(\frac{\partial \Delta_1}{\partial S} \right) + n_2 \left(\frac{\partial \Delta_2}{\partial S} \right), \\ \Gamma &= n_1 \Gamma_1 + n_2 \Gamma_2. \end{aligned} \tag{7}$$

► *Knowing a portfolio's gamma and delta positions can be helpful in defining an investor's risk-return exposure.* For example, suppose a delta neutral portfolio is formed. If this portfolio has a positive gamma, then the portfolio will decline in value if there is little or no change in the stock price and increase in value if there is a large positive or negative change in the stock price. On the other hand, if the delta-neutral portfolio has a negative gamma, the portfolio will increase in value if there is little or no change in the stock price and decrease in value if there is a large positive or negative change in the stock price. (These relations are depicted in Figure 6.)

- The **Position Theta** (Θ_P) is given by the partial derivative of equation (4) with respect to the time to expiration T :

$$\begin{aligned} \Theta_P &= \frac{\partial V}{\partial T} = n_1 \left(\frac{\partial C_1}{\partial T} \right) + n_2 \left(\frac{\partial C_2}{\partial T} \right), \text{ or} \\ \Theta_P &= n_1 \Theta_1 + n_2 \Theta_2. \end{aligned} \tag{8}$$

  In general, the larger the position's theta, the greater its time value decay. Thus, taking a short position in an option portfolio with a large time value decay may be a profitable strategy, especially if the position has low delta and gamma values.