Brunel University
Msc., EC5504, Financial Engineering
Prof Menelaos Karanasos

Lecture Notes: Basic Concepts in Option Pricing - The Black and Scholes Model (Continued)

- In previous lectures we saw that the intrinsic value of an option is determined by taking the difference between the market price and the discounted strike price. Volatility, time to expiration, and our normal distribution function allow us to determine the time value component.[1] Figure 1.1 shows the relative contributions of the time value and intrinsic value to the total value of a call option. Figure 1.2 displays prices generated by the Black and Scholes model for call options. If we combine the intrinsic and the time value components, graphed in Figure 1.1, we will achieve the same result as in Figure 1.2. After the addition of the time value, the relationship between the option's premium and the underlying price is represented by a smooth curve. Prior to expiration, a call option's relationship to the underlying market is defined by this curve instead of the angular "hockey stick" shape of the (intrinsic value) profit/loss profiles. The longer the period until expiration, the greater the price for an option and the straighter the shape of the curve. Only at expiration, when there is no time remaining, does the call option's profit/loss profile resemble a "hockey stick" shape and the price of the option is only determined by the prices of the underlying asset and the strike.

Despite its severe assumptions (e.g. constant volatility and interest rates over time, no cash flows associated with the underlying asset), the Black and Scholes model is remarkably good as an estimation tool for the "fair" price of an option. Essentially, Black and Scholes required these assumptions to take a "snap shot" of what the options price would be at a single point in time. At the moment of estimation, the BS formula is as accurate as a Nikkon camera. It will give us a "fair" picture of the situation. If, however, the scene changes, the photograph we have taken is no longer an accurate representation of current reality. As with the BS model, one must retake

[1] Note that when the option is at-the-money, the intrinsic value of the option is zero and thus time value is at its peak. When the stock price is lower or higher than the strike price time value will fall. Ultimately, at some level of the underlying stock price the time value will approach zero and the option will either be deeply in- or out-of-the money.

the picture when the scene changes. Does that imply that a Nikkon camera is not good simply because the scenes it records change? Obviously not, and equally so that does not mean that the BS model is invalid because it assumes the underlying market conditions will remain unchanged. We should remember that the BS formula is referred to as a model and not as the Black and Scholes reality.

- Recall that the BS formula is given by

  $$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2), \text{ where } d_2 = d_1 - \sigma\sqrt{T}.$$

  Observe that at expiration $T = 0$ and so the factors $d_{1,}d_2$ are identical. As a result, the output from the normal distribution is the same number, let's say $N(d)$. So we can write:

  $$C_0 = \left(S_0 - Xe^{-rT}\right) N(d).$$

  It is easy to see that when $N(d) = 1$ the value of the option equals the intrinsic value. On the other hand, if $N(d) = 0$ the value of the option is simply zero. This is the result that one finds at expiration.[2] The option worths either its positive intrinsic value or zero. However, before expiration, d_1 and d_2 are not identical. The difference between these factors is the **key element which determines the time value of an option**:

  $$\sigma\sqrt{T},$$

  where σ is the volatility of the underlying stock, and T is the time to expiration of the option in the percentage of a year.

▼ Effects of Time on the Price of an Option ▼

Since we know that options are similar to insurance, we can use an insurance example to infer how time affects the value of an option. If we compare a one year insurance policy to a two year insurance policy, both starting tomorrow, we would expect them to have different prices. The longer the term of the policy is, the more expensive the insurance will be. Similarly for options, the greater the time to expiration the higher its time value. However,

[2] At expiration $T = 0$. If $S > X$ then $\lim_{T\to 0} d = +\infty$, and so $N(d) = 1$. If $S < X$ then $\lim_{T\to 0} d = -\infty$, and so $N(d) = 0$.

time value is not a linear function of time.[3] The impact of time is a function of the square root of the time to expiration. Because of this fact, one might conclude that there are optimal times to consider purchasing options, and times when it is optimal to sell them, purely from the aspect of time decay.

For example, assume you have two choices for purchasing 125 days of option protection: The first choice would be to buy an at-the-money IBM call option with 125 days left until expiration for \$6 and hold it until it expires for a total cost (loss) of $6 \times 100 = \$600$. (Note that the option contract is for 100 shares). The second choice is to buy an at-the-money IBM call option with 250 days until expiration for \$8.5, and sell it after 125 days for \$6, with a total cost of only $2.5 \times 100 = 250$. The second choice is cheaper than the first, given that the other variables which affect an options price remain constant. (See Figure 2.1.)

- The time to consider buying options *from a time decay standpoint* is when the option has a long term until expiration. The time to consider selling options is when the option has a short term until expiration.

If one recognizes this fact, then why anyone buy options close to the expiration date? This is especially puzzling when one considers that the most concentrated trading activity in exchange traded options seems to occur in those options closest to expiration. Obviously that means that a lot of people are buying options at a time where the heaviest rate of time decay occurs. To address this puzzle we have to consider another factor: **leverage.**

Suppose there are three kinds of players in the market: hedgers, market makers and speculators. **Hedgers** are interested in reducing their risks. Therefore they are more likely to buy long dated options and sell them back prior to expiration to reduce the expected cost. Most economists would categorize hedgers as *risk averse*. **Market makers** should be indifferent about when they buy or sell options because they generally hold positions only long enough to find someone else to pass the contracts on to. Therefore, they might be viewed as *risk neutral* regarding the time decay of options.

[3] In particular, we can say that time value as a function of time to expiration is increasing at a decreasing rate:

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\begin{aligned}
\frac{\partial\left(\sigma\sqrt{T}\right)}{\partial T} &= \frac{\sigma}{2\sqrt{T}} > 0, \\
\frac{\partial^2\left(\sigma\sqrt{T}\right)}{\partial T^2} &= -\frac{\sigma}{4T^{3/2}} < 0.
\end{aligned}
$$

Speculators are interested in making a quick profit; therefore they seek to maximize the amount of leverage (percentage returns) and they are *risk seekers.*

Figure 2.2 shows that leverage is maximized as options approach expiration. In this figure leverage is defined as the percentage profit for an at-the-money IBM option for a fixed \$5 movement in the underlying market at various points prior to expiration. The further back in time one goes, the greater the option's cost and the lower the leverage. However, if one does purchase a long dated option, the time decay impact is significant less. Hedgers, who often have more capital to work with, want to minimize the net cost of their option premium. They can achieve this by paying more for the option initially and then offsetting them before the heavy time decay begins to occur.

Why would a speculator prefer inexpensive options near the expiration date? Suppose the speculator considers buying an option with 200 days until expiration costing \$8, or an option with approximately a week of life left that costs only \$0.25. If the underlying stock price increases by \$5, then the long dated option's value would increase to \$10.5 for the \$5 increase in the price of the underlying share. This is only a 31% return on the investment of the premium.[4] Now, consider the short dated option bought at \$0.25. When the market moves by \$5, then the option's value would increase to \$5. The percentage return is then 1,900%.[5]

- The leverage is inversely related to the time remaining and the shape of the leverage curve is the "mirror image" of the time decay curve.

Speculators can make a huge percentage return by buying options with strike prices which are close to the prevailing market price. For example, with one day to expiration, someone buys an IBM \$65 call for \$0.25 when the underlying stock price is \$64.75. If IBM's stock price happened to move to \$65.5 the next day, the holder would exercise the option, sell the stock and make a profit of \$0.25. This would be a return of 100% in one day on the capital invested. Therefore, there exists an incentive to manipulate the underlying market near expiration to drive the market price through an

4

$$\text{Leverage} = \frac{10.5 - 8}{8} = 0.3125.$$

5

$$\text{Leverage} = \frac{5 - 0.25}{0.25} = 19.00.$$

option strike price. (Note that many exchanges have very strict regulations on trading near expiration.) Because of the highly levered, limited loss gamble that exists during the last week before expiration, many option professionals have coined this period "lottery time".

▼ The Lognormal Distribution and the Time Value of an Option ▼

Since Black and Scholes assume that the markets are distributed lognormally, the highest probability event is at the current market price and this is the best guess for the outcome of the next market price. Thus, **at-the-money bets have the highest probability and, as a result, the highest time value.**When the option's strike price is deeply out-of-the money, the option is a long shot and not very expensive. If the option is deeply in-the-money, it will be very expensive. One reduces the leverage and potential profits of the option because of the high premium expenditure. So, from a net profit standpoint, the lowest expected profits are for deeply out-of-the money and deeply in-the-money options. The greatest profit potential is for at-the-money options.

▼ The Impact of Volatility on the Time Value of an Option ▼

The other major component in determining time value is volatility. The volatility of a stock (σ) is a measure of our uncertainty about the returns provided by the stock. The more volatile the underlying market, the riskier it is. Volatility is by far the most important variable in the determination of the time value of an option premium. The greater the risk, the higher the option premium. Looking at the Black and Scholes model, remember that: $d_2 = d_1 - \sigma\sqrt{T}$. The only difference between these two factors and the key element which determines the time value of an option is $\sigma\sqrt{T}$. Whereas the square root of time is involved for time decay, the impact on an option from a change in volatility parameter is directly proportional.

- The higher the volatility, the bigger the difference between the factors d_1 and d_2 and the higher the time value of an option.
- The volatility of a stock can be defined as the standard deviation of the return provided by the stock in one year when the return is expressed using continuous compounding.

► Historical Volatility We estimate the volatility of a stock price in the following way. We obtain a time series of past prices:

$$P_t, \text{ where } t = 1, 2, ..., T+1.$$

This is a data set of $T+1$ observations on price. We define the rate of return on the stock:

$$R_t = \ln P_t - \ln P_{t-1}, \text{ where } t = 2, 3, ..., T+1.$$

This is a set of T data points. We then calculate the sample variance (s^2) of the rate of return:

$$\begin{aligned} s^2 &= \frac{1}{T-1}\sum_{t=2}^{T+1}\left(R_t - \overline{R}\right)^2 \\ &= \frac{1}{T-1}\sum_{t=1}^{T+1} R_t^2 - \frac{1}{T(T-1)}\left(\sum_{t=1}^{T+1} R_t\right)^2, \end{aligned}$$

where $\overline{R} = \frac{1}{T}\sum_{t=2}^{T+1} R_t$ is the sample mean of the rate of return. Finally, to annualize the above variance we multiply it by the frequency of our sampling, i.e.

$$\begin{aligned} \text{for quarterly data } \sigma^2 &= s^2(4), \ \sigma = s\sqrt{4}; \\ \text{for monthly data } \sigma^2 &= s^2(12), \ \sigma = s\sqrt{12}; \\ \text{for weekly data } \sigma^2 &= s^2(52) \ \sigma = s\sqrt{52}; \\ \text{for daily data } \sigma^2 &= s^2(252), \ \sigma = s\sqrt{252}. \end{aligned}$$

(Note that in the later formula we have used the number of trading days as opposed to calendar days.)

It can be shown that the standard error of the above estimate is given by

$$(\text{s.e. of } \sigma) : \frac{\sigma}{\sqrt{2T}}.$$

There is no clear cut answer as to the appropriate sample size (T) to be used in the above estimation. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days. An often used rule of thumb is to set the time period over which the volatility is measured equal to the time period over which to be applied. For example, if the volatility is to be used to value a two-year option, two years of historical data are used.

It is also important to note that the above estimation of volatility assumes the following random walk model for log prices:[6]

$$\begin{aligned} \ln P_t &= \ln P_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim IIN(0,h), \\ \text{i.e. } R_t &= \varepsilon_t. \end{aligned}$$

In other words we assume that the conditional variance of the rate of return (h) is constant. If this assumption does not hold then we can use GARCH (Generalized Autoregressive Conditional Heteroscedasticity) models to estimate the time varying volatility. For example, consider the GARCH(1,1) model given below:

$$\begin{aligned} R_t &= \varepsilon_t, \text{ where } \varepsilon_t \mid \varepsilon_{t-1} \sim IIN(0,h_t) \text{ and} \\ h_t &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}. \end{aligned}$$

GARCH models are beyond the scope of this course.

► Implied Volatility This is the volatility implied by an option price observed in the market. To illustrate the basic idea, suppose that for a call on a non-dividend-paying stock we have: $C_0 = 1.875$, $S_0 = 21$, $X = 20$, $r = 0.1$, $T = 0.25$. The implied volatility is the value of σ, that when substituted into the BS formula gives $C_0 = 1.875$. Unfortunately, it is not possible to invert the BS equation so that σ is expressed as a function of S_0, X, r, T, and C_0. So we need to use an iterative search procedure to find the implied σ. In this example, the implied volatility is 23.5% per annum.

[6] Without loss of generality we use a random walk model without drift.