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Lecture Notes: Basic Concepts in Option Pricing - The Black and Scholes Model

Recall that the price of an option is equal to the intrinsic value of the option plus its time value. Since the intrinsic value is simply the in-the-money amount, it is easy to see why the intrinsic value component of an option price is trivial to estimate. The problem with option pricing lies with the time value. Fischer Black and Myron Scholes made a breakthrough in options pricing in 1972-73 when they "cracked the nut" of time value.[1]

- Before proceeding with the presentation and explanation of the Black and Scholes model we will introduce stochastic dominance arguments which are used to define the boundary limits for option prices. The **stochastic dominance** condition[2] is a statement about the relative benefit of different alternative investments. Stochastic means probabilistic or random. Dominance means that one of the two instruments dominates the other in terms of the return that will be provided to the investor.[3] Stochastic dominance arguments are used to define the boundary limits for option prices. Consider that a call option is American style, meaning it can be exercised at any time.

    1. The first stochastic dominance condition compares the option's price to its intrinsic value. *The option's price must be greater than or equal to its intrinsic value.*[4]

[1]Black and Scholes, "The Pricing of Options and Corporate Liabilities", Journal of Political Economy, 1973. While Black and Scholes got all the glory, Robert Merton was hot on the trail of solving the problem and barely missed out. See Merton, "Theory of Rational Option Pricing", Bell Journal of Economics and Management Science, 1973, for one of the best early articles on option pricing.

[2]Note that this is a preference-free condition, i.e. we rank investments according to the expected value and not the volatility of their payoffs.

[3]For example, suppose that we are comparing two alternative investments, A and B, in a market that can go up, down, or remain the same. If investment A provides the same payoff as investment B when the market goes up or down, but provides a superior payoff when the market is stable, then we say that *investment A is stochastically dominant over investment B.*

[4]If the option price were less than the intrinsic value, the option buyer could earn immediate and risk-free profits by purchasing the option, exercising it and covering the exercised position in the underlying market. For example, suppose a $60 call option on

2. The second stochastic dominance condition is that *an option must be worth at least zero.*[5]
3. The third stochastic dominance condition is that an *option price must be less than or equal to the value of the underlying asset price.*[6]

- **Boundary Conditions for Call Option Prices:** the option price must be some positive non-zero value between the intrinsic value and the value of the underlying asset (see Figure 1).

With these stochastic dominance arguments defining the boundaries for call option prices, Black and Scholes determined where in that range the call option price will be. The **assumptions of the Black and Scholes model** are:

(A1) The call option has a European style exercise feature.
(A2) Security trading is continuous.
(A3) The underlying market price is distributed lognormally with constant mean and variance.
(A4) The risk-free rate of interest is constant, and the term structure of interest rates is flat.
(A5) The underlying asset pays no dividends or coupons during the life of the derivative.
(A6) There are no transaction costs or taxes, all securities are perfectly divisible, the short selling of securities with full use of proceeds is permitted, and there are no arbitrage opportunities

IBM was trading at \$2 when the underlying IBM stock was trading at \$65. Then the option buyer could buy the call for \$2, immediately exercise it and simultaneously sell the IBM stock at \$65. The result of these series of transaction is an inflow of \$5 from the stock trades (long IBM stock through the call exercise at \$60 and selling the stock obtained at \$65) and an outflow of \$2 on the options purchase. The net effect is a risk-free inflow of \$3.

[5] Recall that writers (sellers) of options act like insurance agents. Imagine an insurance broker providing you with insurance and not only giving you the insurance for free, but throwing in a cash rebate as well. Likewise with an option, the option price conceivably could be zero if there is no risk, but it can never be negative. An option price must be greater or equal to zero.

[6] For example, consider the relationship between the value of a six month option on gold with the actual price of gold. The option on the gold will only last six months while the gold may last forever. So it is irrational that the option worths more than the underlying asset.

It is evident that normal distributions permeate throughout the world whenever we have uncertain outcomes. Do normal distributions also apply in the financial markets? The range of normally distributed random variables is from positive to negative infinity. This would imply that prices for underlying assets could potentially be below zero. This cannot occur. The solution is to convert the actual price levels into their logarithms. The transformed series has the useful feature of being bounded by zero and the ability to rise to positive infinity. Now it makes sense to assume that log prices are normally distributed. This implies that *prices follow the lognormal distribution, while rates of return are normally distributed.* Unlike the normal distribution, the lognormal distribution is skewed to the right, so that mean median and mode are all different. (See Figure 2.)

The Black and Scholes assumption about how markets tend to behave is similar to the random movement of particles in physics. Therefore, it is not surprising that an important breakthrough in solving the Black and Scholes formula for option pricing was the adaptation of an equation for heat transfer from physics.

To illustrate the comparative logic between the heat transfer equation and option pricing, assume you have a block of metal in a room which is at 20 degrees Celsius. If the block was heated until its temperature reached 200^0C and then allowed to cool, the centre of the block would remain hot for quite some time, but at some point will begin cooling down rapidly. If you plotted the decay of heat over time, you would see a curve similar to that in Figure 3.

This pattern of decay is identical to the *time decay predicted for options by the Black and Scholes option pricing model*. This breakthrough allowed, for the first time, a closed form solution for the pricing of European call options. The theory behind adapting the heat transfer equation to options was that if one could simply **determine the market volatility (the "heat") that exists and the time left until expiration, the time value of an option could be estimated.** Once this is added to the intrinsic value, you have determined the "fair" option price.

- **The Black and Scholes formula for the price at time zero of a European call option** (C_0) is given by

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2), \tag{1}$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \text{ and} \tag{1a}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}, \tag{1b}$$

where S_0 is the price of the underlying asset at time $t = 0$, X is the strike price, T is time to maturity, r is the risk-free interest rate, and σ is the stock price volatility. $N(\cdot)$ denotes the distribution function for a random variable $x \sim N(0,1)$. In other words, $N(d_1)$ is the probability that x is less than or equal to d_1.

▼ Understanding the Black and Scholes Model ▼

The Black and Scholes (BS hereafter) formula is really quite easy to understand when one remembers that the price of an option is simply made up of its intrinsic value and time value. For a call option, the intrinsic value is determined by the difference in the underlying stock price and the exercise price. In the BS equation S_0 is the current price of the stock and Xe^{-rT} is the present value of the exercise price of the option. If the price of the option were only made up of the intrinsic value (as is the case at the expiration of the options) then the Black and Scholes formula simply gives the difference between the stock price and today's equivalent exercise price, i.e., $S_0 - Xe^{rT}$, to yield the (discounted) intrinsic value of the call. (Obviously at expiration, the "future is now" and the intrinsic value is simply $S - X$.)[7]

$$\textbf{Intrinsic (Present) Value of the Call} = S_0 - Xe^{rT}. \tag{2}$$

[7] The exercise price is discounted because the BS formula was derived only for European style options. For the American style option, the fact that immediate exercise could occur means the intrinsic value must be equal to the difference between the current underlying price and the strike price. For the European option, the only time exercise could occur to realize the intrinsic value is at the expiration in the future. To determine today's equivalent exercise price, we must discount the future exercise price back to present value and e^{-rT} is the continuous interest rate factor that achieves this.

The above factor is critical in understanding the relationship between American and European call option prices. Figure 4 provides the stochastic dominance argument for the minimum price of the call option relative to its intrinsic value for both American and European calls. Note that, for non-dividend-paying underlying stocks, *the price of a European call option must be greater than or equal to the price of an American call option.* This may seem counter-intuitive since an American option allows the trader something (an early exercise feature) that the European option does not.

So one would expect an American call to worth more than a European call. We explain why this is not so. The holder of an American call will decide to strike when the option is in-the-money. His payoff is equal to the intrinsic value of the option. However, prior to expiration, the option has also time value. Upon exercise before the final expiration date, this time value is lost. It is comparable to taking money out of your pocket and throwing it away. If one wanted to offset the option rather than exercising it, one could sell it and receive at least as much.

So, for almost all assets, the American exercise feature does not add value to the value of the call option because there is no value in being able to exercise early if you could instead sell the option.[8] This explains why most people choose to offset options rather than exercise them. It also shows that there is no serious problem with the fact that the BS model holds for European calls while most options have an American exercise feature.

By process of elimination, the time value for the Black and Scholes model must be determined by the other factors in eq. (1), i.e. $N(d_1)$ and $N(d_2)$. Let us return to the logic of the process. The two sources of value to the buyer of a call option are:

(i) an unlimited profit potential when the stock price S_T is above X, and

(ii) a limited loss potential when the stock price S_T is below X.

- Clearly, when one is betting on the market using an option, the trader is interested in the probability that the option will be profitable at the end of its life. As with any asset, what **the investor is willing to pay for the option is the present value of the potential future profits.**

To assess the probabilities, some function must be applied to the possible movements of the underlying stock price. Black and Scholes assume that this

[8] This may not be the case for a call option on an asset which pays coupons or dividends. There also exist circumstances where the early exercise of a put option may make sense (see Merton, 1973).

process is that of a normal distribution for the returns of the stock (and a lognormal distribution for the prices).

Let us write the above formally as an equation:

$$C_0 = \text{Prob}\left(S_T > X\right) \cdot \left[E\left(S_T \mid S_0 > X\right) - X\right] \cdot e^{-rT}, \tag{3}$$

where $E\left(S_T \mid S_0 > X\right)$ denotes the expectation (conditional upon current information) of a variable that equals S_T if $S_T > X$ and is zero otherwise.

The above equation says that we are interested in the expected value of the future payoffs: $\left[E\left(S_T \mid S_0 > X\right) - X\right]$.

Then, this value is discounted back to present value: $\left[E\left(S_T \mid S > X\right) - X\right] \cdot e^{-rT}$.

Finally, since the payoff is an uncertain outcome, we need to weight it by the probability of its occurence.

It can be shown that

$$E\left(S_T \mid S_0 > X\right) = S_0 \frac{N\left(d_1\right)}{N\left(d_2\right)} e^{rT}, \tag{3a}$$

where $N\left(d_1\right), N\left(d_2\right)$ have been defined in equations (1a)-(1b).

It can also be shown that the probability that the option will be exercised, i.e. Prob$(S_T > X)$, is given by $N\left(d_2\right)$:

$$\text{Prob}\left(S_T > X\right) = N\left(d_2\right). \tag{3b}$$

Now substitute eq.(3a)-(3b) into eq. (3) to get:

$$\begin{aligned} C_0 &= N\left(d_2\right) \cdot \left[S_0 \frac{N\left(d_1\right)}{N\left(d_2\right)} e^{rT} - X\right] \cdot e^{-rT} \\ &= S_0 N\left(d_1\right) - X e^{-rT} N\left(d_2\right). \end{aligned}$$

Note that the latter is the BS equation (1).[9]

[9] The Black and Scholes model (1) also gives the premium for an American call option on a non-dividend-paying stock. Note that no exact analytic formula for the value of an American put option on a non-dividend-paying stock has been produced. American put options are being valued using numerical procedures and analytical approximations.

► We now derive an important relationship between the price of a European put $\left(P^E\right)$ and a European call $\left(C^E\right)$ for a non-dividend-paying stock. Consider the following two portfolios.

- – Portfolio A: one European call option (with strike price X, and expiration date T) plus an amount of cash equal to Xe^{-rT}.
  – Portfolio B: one European put option (with strike price X, and expiration date T) plus one share.

At expiration of the options, both are worth

$$\max\left(S_T,\ X\right).$$

Because the options are European, they cannot be exercised prior to the expiration date. Therefore, the portfolios must have identical values today.

- The fact that portfolios A and B must have identical values means that

$$C^E + Xe^{-rT} = P^E + S_0. \tag{4}$$

  This relationship is known as **put-call parity**. It shows that the value of a European put with a certain exercise price and exercise date can be deduced from the value of a European put with the same exercise price and date, and vice versa. If eq. (4) does not hold, there are arbitrage opportunities (for an example see the relevant exercise in problem set 2).

- **Factors affecting the price of a stock option:**

1. The current stock price (S_t).
2. The strike price (X).
3. The time to expiration (T).
4. The volatility of the stock price (σ).
5. The risk-free interest rate (r).
(6. The dividends expected during the life of the option.)

The following table summarizes what happens to option prices when one of the above factors changes with all the others remaining fixed.

Variable	European Call	European Put	American Call	American Put
Stock price	+	−	+	−
Strike price	−	+	−	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free rate	+	−	+	−
Dividends	−	+	−	+