Duration of Bonds

Professor Menelaos Karanasos

December 2011

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Consider three statements that concern **coupon-bearing bonds:**

- The prices of bonds with equal maturities are equally sensitive to changes in interest rates.
- The sensitivity of the price of the bond to changes in interest rates increases with the maturity of the bond.
- If an investor buys a bond whose maturity matches the date on which he requires funds, subsequent interest rate movements have no effect on his total return over the investment interval.

These statements, although popularly believed and most true for discount bonds, are not true for coupon-bearing bonds.
They are nearly true, however, if maturity is replaced with **duration.**

A bond's stream of payments consists of
(i) coupon payments during the life of the bond and
(ii) repayment of principal at maturity. For some purposes it is useful to have a measure of the futurity of a bond's cash flow.
Note that maturity refers only to the time remaining to the repayment of principal and disregards earlier coupon payments.
So speaking, term to maturity is not a useful measure of the futurity of a bond's cash flow.

In what follows we will

(**a**) introduce the concept of duration as a measure of the futurity of a bond's payment stream,

(**b**) show how duration is a measure of a bond's interest rate sensitivity, and

(**c**) demonstrate how, given an investment horizon, proper choice of duration can **immunize** the exposure of a portfolio to the risk of interest rate fluctuations over the horizon.

For expositional simplicity we assume that bonds make annual coupon payments and that yields are computed with annual compounding. Consider a bond which has n coupons remaining to be paid where each coupon is for C currency units per 100 currency units principal amount (face value).

The present value of the bond is given by:

$$P = \frac{C}{(1+Y)} + \frac{C}{(1+Y)^2} + ... + \frac{C}{(1+Y)^{n-1}} + \frac{C+100}{(1+Y)^n}. \qquad (1)$$

(We assume that the annual yield is the same for all bonds and equal to Y, i.e., the yield curve is *flat*).

The derivative of the price of the bond with respect to its yield is:

$\frac{dP}{dY} = -\frac{C}{(1+Y)^2} - \frac{2C}{(1+Y)^3} - ... - \frac{(n-1)C}{(1+Y)^n} - \frac{n(C+100)}{(1+Y)^{n+1}}$ (2)

$$\frac{dP}{dY} = -\frac{C}{(1+Y)^2} - \frac{2C}{(1+Y)^3} - \ldots - \frac{(n-1)\,C}{(1+Y)^n} - \frac{n\,(C+100)}{(1+Y)^{n+1}}. \tag{2}$$

After multiplying both sides of the above expression by $(1+Y)$, we obtain:

$$(1+Y)\left(\frac{dP}{dY}\right) = -\left[\frac{C}{(1+Y)} + \ldots + \frac{(n-1)\,C}{(1+Y)^{n-1}} + \frac{n\,(C+100)}{(1+Y)^n}\right]. \tag{3}$$

▶▶▶ To define the duration of the bond, we need to weight the time to each payment by the present value of the payment and then divide the result by the sum of the weights.

This gives a measure of *waiting time corrected for present value discounting.*

Using (1), **duration** is defined as follows:

$$D = \frac{\overbrace{\frac{C}{(1+Y)} + \frac{2C}{(1+Y)^2} + ... + \frac{(n-1)\,C}{(1+Y)^{n-1}} + \frac{n\,(C+100)}{(1+Y)^n}}^{-(1+Y)\left(\frac{dP}{dY}\right)}}{P}, \text{ or}$$

$$D = -\frac{(1+Y)}{P}\left(\frac{dP}{dY}\right) \tag{4}$$

Duration, D, is a weighted average of all the payments periods: $1, 2, \ldots, n$

Each weight is the present value of the payment:

i.e., For period $n-1$: $\frac{C}{(1+Y)^{n-1}}$.

We divide each weight by P, so the sum of all the weights is 1

$$\frac{\frac{C}{(1+Y)}+\frac{C}{(1+Y)^2}+\ldots+\frac{C}{(1+Y)^{n-1}}+\frac{(C+100)}{(1+Y)^n}}{P}=\frac{P}{P}=1$$

Observations:

(i) in the case of a discount bond, $C = 0$, therefore $D = n$: the duration of a zero bond is always and identically equal to its term to maturity;

(ii) increasing the coupon on a bond will reduce the duration of the bond; increasing the yield on a bond will also reduce the duration of the bond;

(iii) duration does not contract on a year-for-year basis as a bond approaches maturity - the bond ages more slowly than time passes (only the duration of a zero declines on a year-for-year basis as it approaches maturity).

►►►In view of (4), the right-hand-side of equation (3) may be written as the product of duration and price:

$$(1+Y)\left(\frac{dP}{dY}\right) = -DP. \tag{5a}$$

We may write the above equation in terms of finite changes:

$$\begin{aligned} \frac{\Delta P}{\Delta Y} &= \frac{-DP}{(1+Y)}, \text{ or} \\ \frac{\Delta P}{P} &= \frac{-D}{(1+Y)}\Delta Y. \end{aligned} \tag{5b}$$

Interpretation of eq. (5b): *the relative change in the price of the bond* $\left[\frac{\Delta P}{P}\right]$ *is proportional to the absolute change in yield* $[\Delta Y]$ *where the factor of proportionality* $\left[\frac{-D}{(1+Y)}\right]$ *is a function of the bond's duration.*

Furthermore, for a given change in yield, longer duration bonds have greater relative price volatility. This implies that anything that causes an increase in a bond's duration serves to raise its **interest rate sensitivity**, and vice-versa.

Therefore, if interest rates are expected to fall, bonds with lower coupons can be expected to appreciate faster than higher coupon bonds of the same maturity.

Zeros should afford the greatest capital appreciation in a rising market. Conversely, zero coupon and low coupon bonds may be relatively less desirable if the market is expected to decline.

Duration as a slope:

$$
\begin{aligned}
D &= \frac{-dP}{dY}\frac{(1+Y)}{P} \\
&= \underbrace{\frac{-dP}{d(1+Y)}}_{dY}\frac{(1+Y)}{P} \\
&= e^{P}_{1+y} = \frac{d\ln(P)}{d\ln(1+P)}.
\end{aligned}
$$

▶▶▶Next, we examine **duration as a balance point between reinvestment risk and market risk.**

Consider an investor with an investment horizon of h years (i.e. the s/he needs the money back in exactly h years) and a bond that has n annual coupons remaining to be paid.

Let us examine the position of the investor at the end of his h-year investment horizon, and assume that m coupons are received before the horizon date and that $(n - m)$ coupons remain to be paid as of the horizon date.

The investor's final wealth can be divided into the following two components.

- The proceeds of the m coupons received before the horizon date and reinvested (at interest rate Y) to that date:

$$I = C(1+Y)^{h-1} + C(1+Y)^{h-2} + \ldots + C(1+Y)^{h-m}, \quad (6)$$

$$\text{where } \frac{\Delta I}{\Delta Y} > 0. \quad (7)$$

The above shows that an investor who buys a bond with maturity less than (or equal to) her investment horizon

is exposed to ***reinvestment risk***: if interest rates go up (down) the investor is better off (worse off).

- The liquidation value of the unmatured bond on the horizon date which can be derived by discounting the remaining cash flows on the bond back to that date:

$$L = \frac{C}{(1+Y)^{m+1-h}} + \ldots + \tag{8}$$

$$+\frac{C}{(1+Y)^{n-1-h}} + \frac{C+100}{(1+Y)^{n-h}},$$

$$\text{where } \frac{\Delta L}{\Delta Y} < 0. \tag{9}$$

The above shows that an investor who buys a bond with maturity more than his investment horizon

is exposed to *market risk*: if interest rates go up (down) the investor is worse off (better off).
This kind of risk can be thought of as the reciprocal of reinvestment risk.

Both of these components depend on the level of interest rates following purchase of the bond.

If, for example, rates rise, then the first component will be larger, and the second component will be smaller.

The bond as a whole is an *immunizing* or riskless investment if any decrease in one component is exactly offset by an increase in the other component.

For simplicity we assume that market yields can change only once, immediately after the purchase of the bond.

The investor's final wealth is the sum of the two components identified above, i.e.:

$$W = I + L. \tag{10}$$

We want to find out if it is possible to have a bond such that $\frac{\Delta W}{\Delta Y} = 0$, that is, such that the final wealth is invariant with respect to changes in interest rates.

Combining (6) and (8) in (10), we obtain:

$$W = \underbrace{C(1+Y)^{h-1} + ... + C(1+Y)^{h-m}}_{I} + \overbrace{\frac{C}{(1+Y)^{m+1-h}} + ... + \frac{C+100}{(1+Y)^{n-h}}}^{L},$$

which after dividing both sides by $(1+Y)^h$ gives:

$$\frac{W}{(1+Y)^h} = \frac{C}{(1+Y)} + ... + \frac{C}{(1+Y)^m} + \frac{C}{(1+Y)^{m+1}} + ... + \frac{C+100}{(1+Y)^n}. \quad (11)$$

The term on the right-hand-side of eq. (11) is just the price (P) of the bond at the beginning of the investment period when the yield on the bond is Y. So we have that:

$$W = (1+Y)^h P.$$

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The change in W for a given change in Y is given by:

$$\begin{aligned} \frac{dW}{dY} &= \overbrace{h(1+Y)^{h-1}}^{\frac{d(1+Y)^h}{dh}} P + (1+Y)^h \left(\frac{dP}{dY}\right) \\ &= (1+Y)^h \left[\frac{dP}{dY} + \frac{hP}{(1+Y)}\right]. \end{aligned}$$

Substituting the duration expression (5b) in the above equation, we obtain:

$$\frac{dW}{dY} = (1+Y)^h \left[\overbrace{\frac{-DP}{(1+Y)}}^{\frac{dP}{dY}} + \frac{hP}{(1+Y)}\right], \text{ or} \tag{12a}$$

$$\frac{dW}{dY} = P(1+Y)^{h-1}(h-D). \tag{12b}$$

Observations: (i) the investor's final wealth does not depend on Y if the duration of the bond matches the investor's horizon,
that is, $\frac{dW}{dY} = 0$ if $D = h$;
(ii) the bond exposes the investor to market risk if the duration of the bond exceeds his investment horizon,
that is , $\frac{dW}{dY} < 0$ if $D > h$;
(iii) the bond exposes the investor to reinvestment risk if the duration of the bond is shorter than his investment horizon,
that is, $\frac{dW}{dY} > 0$ if $D < h$.

A bond is said to immunize an investor against interest rate fluctuations if its *duration is just equal to the investor's horizon,*
that is, *reinvestment risk balances market risk when duration equals investment horizon.*
Since duration depends on yield, immunizing bond portfolios have to be *rebalanced* from time to time if interest rates change significantly.