BLACK AND SCHOLES OPTION PRICING MODEL

Assumptions of the model:

1. We will only examine European options.

That is, options that can be exercised only at expiration

2. There are no transaction costs.

In other words, information is available to all without cost

3. The short-term interest rate (r) is known and constant.

Market participants can both borrow and lend at this rate

4. Stocks pay no dividends

5. We are in continuous time

6. The probability distribution of stock returns is normal

7. The variance of the return is constant over the life of the option contract and is known to market participants

(σ_R^2)

BLACK AND SCHOLES (BS) FORMULA

The equilibrium price of the call option (C) is shown by Black and Scholes to be:

$$C = S_t N(d_1) - X e^{-r(T-t)} N(d_2),$$

where S_t is the current price of the stock;

$N(d)$ is the cumulative normal probability density function (see figure 1)

X is the exercise price of the call; r is the short-term interest rate; $e = 2.71828$

Moreover d_1 and d_2 are given by

$$d_1 = \frac{\ln(\frac{S_t}{X}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(\frac{S_t}{X}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Note that $d_2 = d_1 - \sigma\sqrt{T}$

where σ^2 is the variance of the annual rate of return on the stock (continuously compounded)

INTERPRETING THE BS FORMULA

A. $N(d_1)$ represents the delta, or hedge ratio of shares of stocks to options necessary to maintain a fully hedged position

B. The option holder can be viewed as a leveled investor

He or she borrows an amount equal to the present value of X times an adjustment factor $N(d_2)$

Price of a call option:

$$\underbrace{N(d_1)}_{\text{Option Delta}}\underbrace{S_t}_{\text{Stock Price}} - \underbrace{\underbrace{N(d_2)}_{\text{Adjusted factor}}\underbrace{Xe^{-r(T-t)}}_{\text{Present Value of } X}}_{\text{Loan adjusted}}$$

C. The important implication of the Black-Scholes model is that the value of an a option (its equilibrium price) is a function of

i) the current price of the stock (S_t)

ii) the exercise price of the call (X)

iii) the short-term interest rate (r)

iv) the time to expiration (T),

v) the variance of the rate of return of the stock (σ^2)

but it is not a function of the expected return on the stock

LIMITS OF THE BS FORMULA

1. Note that as $T \to \infty$, then $d_1, d_2 \to \infty$.

Thus $N(d_1), N(d_2) \to 1$.

In addition $Xe^{-r(T-t)} \to 0$.

From the above it follows that $C \to S_t$

2. Similarly as $r \to \infty$ then i) $d_1, d_2 \to \infty$,

and therefore $N(d_1), N(d_2) \to 1$.

and ii) $Xe^{-r(T-t)} \to 0$.

i) and ii) imply that $C \to S_t$

3. Finally, as $\sigma \to \infty$: $d_1 \to \infty$ whereas $d_2 \to -\infty$.

Thus $N(d_1) \to 1$ whereas $N(d_2) \to 0$

Therefore $C \to S_t$

Recall that S_t is the upper limit of the call price: $C \leq S_t$

Estimate of σ^2

In solving the formula, we know four of the five variables:

$$S_t, \; X, \; r, \; T$$

The key unknown then is the variance of the stock price, σ^2

The usual approach to the problem is to use the recent past volatility of the stock return as a proxy for its volatility during the life of the option:

$$\begin{bmatrix} 1 \\ 2 \\ \vdots \\ T \end{bmatrix}, \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_T \end{bmatrix}, \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_T \end{bmatrix},$$

where $R_t = \ln(S_t) - \ln(S_{t-1})$

Finally

$$\widehat{\sigma}^2 = \frac{\sum_{t=1}^{T}(R_t - \overline{R})^2}{(T-1)}$$

NUMERICAL EXAMPLE

$S_t = 30;\ X = 28;\ r = 0.1 = 10\%;\ T = 0.5 = 1/2;$
$\sigma = 0.4$

First we calculate

$$
\begin{aligned}
d_1 &= \frac{\ln(\frac{S_t}{X}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\[2mm]
&= \frac{\ln(\frac{30}{28}) + (0.1 + \frac{1}{2}0.4^2)0.5}{0.4\sqrt{0.5}} = 0.562,
\end{aligned}
$$

$$
\begin{aligned}
d_2 &= \frac{\ln(\frac{S_t}{X}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\[2mm]
&= \frac{\ln(\frac{30}{28}) + (0.1 - \frac{1}{2}0.4^2)0.5}{0.4\sqrt{0.5}} = 0.279
\end{aligned}
$$

We will show that $N(d_1) = N(0.562) = 0.713$

Note that: $0.55 < 0.562 < 0.60$

From the table of the normal distribution we have

$$
\begin{aligned}
1 - N(0.55) &= 0.2912, \\
1 - N(0.60) &= 0.2743
\end{aligned}
$$

Using the interpolation method we obtain

$$
\begin{aligned}
1 - N(0.562) &= [1 - N(0.55)] \\
&- \frac{\overbrace{0.012}^{0.562-0.550}}{0.05}\{[1 - N(0.55)] - [1 - N(0.60)]\} \\
&= 0.2912 - \frac{0.012}{0.05}(0.2912 - 0.2743) = 0.287
\end{aligned}
$$

Finally, $1 - N(0.562) = 0.287 \Rightarrow N(0.562) = 1 - 0.287 = 0.713$

Similarly

$$1 - N(0.279) = [1 - N(0.250)]$$

$$-\frac{\overbrace{0.029}^{0.562-0.550}}{0.05}\{[1 - N(0.250)] - [1 - N(0.300)]\}$$

$$= 0.4013 - \frac{0.012}{0.05}(0.4013 - 0.3821) = 0.390$$

Finally, $1 - N(0.279) = 0.390 \Rightarrow N(0.279) = 1 - 0.390 = 0.610$

EQUILIBRIUM PRICES OF CALL AND PUTS

The equilibrium price of the call is

$$
\begin{aligned}
C &= S_t N(d_1) - X e^{-r(T-t)} \\
&= 30(0.713) - 28 e^{-0.1(0.5)} \\
&= 5.14
\end{aligned}
$$

Further, the equilibrium price of the put is given by

$$
\begin{aligned}
P_t &= C + X e^{-r(T-t)} - S_t \\
&= 5.14 + 28 e^{-0.1(0.5)} - 30
\end{aligned}
$$