REVIEW (BINOMIAL MODEL)

- Define a 1 period binomial model
- Construct a portfolio: buy a number of stocks and an amount of money
- Choose the number of stocks and the amount of money in order to replicate the possible payoffs of the call
- Derive the equilibrium price of the call
- Examine the factors that affecting the equilibrium price of the call

- Define a 2 period binomial model

- Derive the equilibrium price of the call for this model

- Derive the equilibrium price of a call for an 3 period binomial model

- Write down the equilibrium price of a call for an n period binomial model

BINOMIAL OPTION PRICING MODEL

Assumptions and Notation

The price of the stock at period zero is denoted by S_0

There are only two periods: 0 and 1.

The are only two possibilities.

The price of the stock in period one can either go up to uS_0, $u > 1$, or down to dS_0, $d < 1$

The price of a call option at period zero is denoted by C_0

The call option expires at period one. There are only two possible states

The two payoffs of the call option are denoted by C_u (in the up state) and C_d (in the down state):

$$C_u = \max(uS_0 - X, 0),\ C_d = \max(dS_0 - X, 0)$$

We have a risk free security. Its return is denoted by R_f

We assume that

$$d < r_f = R_f + 1 < u$$

METHODOLOGY

PORTFOLIO

1. Form a portfolio. In particular, i) buy an amount H_0 of stocks S_0, and ii) borrow an amount of money B_0

The value of this portfolio in period zero is denoted by V_0

In period one the payoff from this portfolio is

$$\begin{aligned} \text{up state} &: \quad H_0 u S_0 - r_f B_0 \\ \text{down state} &: \quad H_0 d S_0 - r_f B_0 \end{aligned}$$

REPLICATE CALL'S PAYOFFS

2 We can choose H_0 and B_0 in such a way in order to replicate the two call's outflows at the end of period 1:

$$\begin{aligned} \text{up state} &: \quad H_0 u S_0 - r_f B_0 = C_u, & (1) \\ \text{down state} &: \quad H_0 d S_0 - r_f B_0 = C_d & (2) \end{aligned}$$

From equation (1) we subtract equation (2):

$$\begin{aligned} H_0 u S_0 - H_0 d S_0 &= C_u - C_d \Rightarrow \\ H_0 S_0 (u - d) &= C_u - C_d \Rightarrow \\ H_0 &= \frac{C_u - C_d}{S_0 (u - d)} \quad (3) \end{aligned}$$

Thus in equilibrium H_0 is equal to the following ratio: spread in the payoffs of the call over the spread in the price of the stock (in the up and down state)

In other words it is the ratio of the range in possible call values to stock values, often referred to as the hedge ratio or delta value

$$H_0 = \frac{C_u - C_d}{S_0(u-d)}$$

Next, substituting equation (3) into either (1) or (2) gives

$$\begin{aligned}
r_f B_0 &= H_0 u S_0 - C_u = u S_0 \frac{C_u - C_d}{S_0(u-d)} - C_u \\
&= \frac{u(C_u - C_d) - C_u(u-d)}{u-d} = \frac{C_u d - C_d u}{u-d} \\
&\Rightarrow B_0 = \frac{C_u d - C_d u}{r_f(u-d)} \qquad (4)
\end{aligned}$$

PRICE OF CALL=VALUE OF PORTFOLIO IN PERIOD 0

$$H_0 = \frac{C_u - C_d}{S_0(u-d)}, B_0 = \frac{C_u d - C_d u}{r_f(u-d)}$$

3. Finally by the law of one price you can determine the equilibrium price of the call, C_0, by setting the current call value equal to the current value of the replicating portfolio

That is (using the above equation)

$$\begin{aligned}
C_0 &= V_0 = H_0 S_0 - B_0 \\
&= S_0 \frac{C_u - C_d}{S_0(u-d)} - \frac{C_u d - C_d u}{r_f(u-d)} \\
&= \frac{1}{r_f(u-d)}[r_f(C_u - C_d) - C_u d + C_d u] \\
&= \frac{1}{r_f(u-d)}[C_u(r_f - d) + C_d(u - r_f)]
\end{aligned}$$

$$C_0 = \frac{1}{r_f(u-d)}[C_u(r_f - d) + C_d(u - r_f)]$$

Next we can denote

$$\frac{r_f - d}{u - d} = p < 1, \frac{u - r_f}{u - d} = 1 - p < 1,$$

and get

$$C_0 = \frac{1}{r_f}[C_u p + C_d(1-p)]$$

In other words, the equilibrium price of the call in period 0 is equal to the present value (we divide by $r_f = 1 + R_f$)

of a weighted average of the two possible payoffs of the call in period 1:C_u and C_d

The weights are given by p and $1 - p$.

SPECIAL CASE

As an example examine the special case where

$$\begin{aligned} uS_0 &> X \Rightarrow C_u = uS_0 - X, \\ dS_0 &< X \Rightarrow C_d = 0 \end{aligned}$$

If this is the case then

$$C_0 = \frac{1}{r_f} C_u p = \frac{(uS_0 - X)}{r_f} (\frac{r - d}{u - d})$$

FACTORS AFFECTING THE PRICE OF CALL

$$C_0 = \frac{(uS_0 - X)}{r_f}\left(\frac{r_f - d}{u - d}\right) \tag{5}$$

From the above equation it follows that

i) As $X \uparrow \rightarrow C_0 \downarrow$

ii) As $S_0 \uparrow \rightarrow C_0 \uparrow$

iii) As $r_f \uparrow \longrightarrow C_0 \uparrow$: since $C_0 = \frac{uS_0 - X}{u-d} - \frac{d(uS_0 - X)}{r_f(u-d)}$

STOCK VOLATILITY

The variance of the stock (σ^2) is given by

$$\sigma^2 = \mathrm{Pr}(u)[uS_0 - E(S)]^2 + \mathrm{Pr}(d)[dS_0 - E(S)]^2$$

where $\mathrm{Pr}(u)$ denotes the probability of being in the up state and $E(S)$ is the expected value of the stock in period 1

The former is 0.5 and the latter is given by

$$\begin{aligned} E(S) &= \mathrm{Pr}(u)uS_0 + \mathrm{Pr}(d)dS_0 \\ &= \frac{S_0}{2}(u + d) \end{aligned}$$

Thus

$$\begin{aligned}\sigma^2 &= \frac{1}{2}[uS_0 - \frac{S_0}{2}(u+d)]^2 + \frac{1}{2}[dS_0 - \frac{S_0}{2}(u+d)]^2 \\ &= S_0^2(\frac{u-d}{2})^2\end{aligned}$$

Thus as either $u \uparrow$ or $d \downarrow \rightarrow \sigma^2 \uparrow$

Finally, note that from equation (5) it follows that as $u \uparrow \longrightarrow C_0 \uparrow$ by $S_0(r_f - d)$ (numerator) and $u \uparrow \longrightarrow C_0 \downarrow$ by r_f (denominator). But since $S_0(r_f - d) > r_f$, the overall effect is positive

2 PERIOD BINOMIAL MODEL

Next we assume that we have two periods. In period two there are three possible values for the price of the stock: $u^2 S_0 = S_{uu}$, $udS_0 = S_{du}$ and $d^2 S_0 = S_{dd}$

Periods		
0	1	2
		$u^2 S_0 = S_{uu}$
S_0	$uS_0 = S_u$	$udS_0 = S_{du}$
	$dS_0 = S_d$	$d^2 S_0 = S_{dd}$

Accordingly, in the second period there are also three possible payoffs for the call: $C_{uu} = \max(S_{uu} - X, 0)$, $C_{ud} = \max(S_{ud} - X, 0)$ and $C_{dd} = \max(S_{dd} - X, 0)$

Periods		
0	1	2
		$C_{uu} = \max(S_{uu} - X, 0)$
C_0	C_u	$C_{ud} = \max(S_{ud} - X, 0)$
	C_d	$C_{dd} = \max(S_{dd} - X, 0)$

1	2
$C_u = \frac{1}{r_f}[C_{uu}p + C_{ud}(1-p)]$	$C_{uu} = \max(S_{uu} - X, 0)$
$C_d = \frac{1}{r_f}[C_{ud}p + C_{dd}(1-p)]$	$C_{ud} = \max(S_{ud} - X, 0)$
	$C_{dd} = \max(S_{dd} - X, 0)$

When the stock price is at the up stage in period 1 we can use the 1 period Binomial model to price the option in period 1 in the up state:

$$C_u = \frac{1}{r_f}[C_{uu}p + C_{ud}(1-p)] \tag{6}$$

Similarly when the stock price is at the down stage in period 1 we can use the 1 period Binomial model to price the option in period 1 in the down state:

$$C_d = \frac{1}{r_f}[C_{ud}p + C_{dd}(1-p)] \tag{7}$$

0	1
$C_0 = \frac{1}{r_f}[C_u p + C_d(1-p)]$	$C_u = \frac{1}{r_f}[C_{uu} p + C_{ud}(1-p)]$ $C_d = \frac{1}{r_f}[C_{ud} p + C_{dd}(1-p)]$

Finally, given the possible call values (C_u and C_d) for period 1 we move to the present and again use the 1 period Binomial model to find C_0:

$$C_0 = \frac{1}{r_f}[C_u p + C_d(1-p)] \tag{8}$$

Substituting equations (6) and (7) into (8) gives:

$$C_0 = \frac{1}{r_f^2}[C_{uu} p^2 + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}]$$

That is

3 PERIODS BINOMIAL MODEL

In period 3 there are four possible payoffs for the call:

$C_{uuu} = \max(S_{uuu} - X, 0)$; $C_{uud} = \max(S_{uud} - X, 0)$;

$C_{udd} = \max(S_{udd} - X, 0)$; $C_{ddd} = \max(S_{ddd} - X, 0)$

Periods

0	1	2	3
			$C_{uuu} = \max(S_{uuu} - X, 0)$
		C_{uu}	
	C_u		$C_{uud} = \max(S_{uud} - X, 0)$
C_0		C_{ud}	
	C_d		$C_{udd} = \max(S_{udd} - X, 0)$
		C_{dd}	
			$C_{ddd} = \max(S_{ddd} - X, 0)$

Steps:

i) We calculate the 3 possible call payoffs in period 2 using the 1 period Binomial model:

$$C_{uu} = \frac{1}{r_f}[C_{uuu}p + C_{uud}(1 - p)];$$

$$C_{ud} = \frac{1}{r_f}[C_{uud}p + C_{udd}(1 - p)];$$

$$C_{dd} = \frac{1}{r_f}[C_{udd}p + C_{ddd}(1 - p)]$$

0	1	2
		C_{uu}
	C_u	
C_0		C_{ud}
	C_d	
		C_{dd}

ii) We calculate the 2 possible call payoffs in period 1 using the 1 period Binomial model:

$$C_u = \frac{1}{r_f}[C_{uu}p + C_{ud}(1-p)];$$

$$C_d = \frac{1}{r_f}[C_{ud}p + C_{dd}(1-p)]$$

iii) We calculate the price of the call in period 0 using the 1 period Binomial model:

$$C_0 = \frac{1}{r_f}[C_u p + C_d(1-p)]$$

Finally, substituting the expressions for C_u, C_d, C_{uu}, C_{ud} and C_{dd} into the expression for C_0 gives

$$\begin{aligned} C_0 &= \frac{1}{r_f^3}[p^3 C_{uuu} + 3p^2(1-p)C_{uud} \\ &\quad +3p(1-p)^2 C_{udd} + (1-p)^3 C_{ddd}] \end{aligned}$$

That is equilibrium price of the call in period 0 is equal to the present value (we divide by r_f^3) of the weighted average of all the 4 possible call payoffs in period 3: $C_{uuu}, C_{uud}, C_{udd}, C_{ddd}$

where the weights can be obtained using the Binomial formula: $\binom{n}{j} p^j (1-p)^{n-j}$ where j is the number of the up states.and the $\binom{n}{j} = \frac{n!}{j!(n-j)!}$

n PERIOD BINOMIAL MODEL

In general if we have an n period model the equilibrium price of the call will be given by

$$\begin{aligned} C_0 &= \frac{1}{r_f^n} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} C_{u^j d^{(n-j)}} \\ &= \frac{1}{r_f^n} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\ &\quad \max[S_0 u^j d^{(n-j)} - X, 0] \end{aligned}$$

If $n = 3$ (we have a 3 period Binomial model) the form will give us

$$\begin{aligned} C_0 &= \frac{1}{r_f^3} [p^3 C_{uuu} + 3p^2(1-p) C_{uud} \\ &\quad + 3p(1-p)^2 C_{udd} + (1-p)^3 C_{ddd}] \end{aligned}$$

since: $\binom{3}{0} = \frac{3!}{0!3!} = 1$, $\binom{3}{1} = \frac{3!}{2!1!} = 3$, $\binom{3}{2} = \frac{3!}{2!1!} = 3$, $\binom{3}{3} = \frac{3!}{3!0!} = 1$.

SUMMARY (BINOMIAL MODEL)

- Defined a 1 period binomial model:

Periods			Periods	
0	1		0	1
S_0	$uS_0 = S_u$ $dS_0 = S_d$	,	C_0	C_u C_d

- Constructed a portfolio: buy a number of stocks and an amount of money:

$$\begin{aligned} \text{up state:} &\quad H_0 u S_0 - r_f B_0 \\ \text{down state} : &\quad H_0 d S_0 - r_f B_0 \end{aligned}$$

- Chosen the number of stocks and the amount of money in order to replicate the possible payoffs of the call:

$$H_0 = \frac{C_u - C_d}{S_0(u-d)}, B_0 = \frac{C_u d - C_d u}{r_f(u-d)}$$

- Derived the equilibrium price of the call:

$$C_0 = \frac{1}{r_f}[C_u p + C_d(1-p)],$$

$$\frac{r_f - d}{u - d} = p < 1, \frac{u - r_f}{u - d} = 1 - p < 1,$$

- Examined the factors that affecting the equilibrium prive of the call:

i) As $X \uparrow \rightarrow C_0 \downarrow$; ii) As $S_0 \uparrow \rightarrow C_0 \uparrow$; iii) As $r_f \uparrow \longrightarrow C_0 \uparrow$; As $\sigma^2 \uparrow \longrightarrow C_0 \uparrow$

- Defined a 2 period binomial model:

$$
\begin{array}{lll}
\text{Periods} & & \\
0 & 1 & 2 \\
 & & C_{uu} = \max(S_{uu} - X, 0) \\
 & C_u & \\
C_0 & & C_{ud} = \max(S_{ud} - X, 0) \\
 & C_d & \\
 & & C_{dd} = \max(S_{dd} - X, 0)
\end{array}
$$

- Derived the equilibrium price of the call

$$C_0 = \frac{1}{r_f^2}[C_{uu}p^2 + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}]$$

- Derive the equilibrium price of a call for an 3 period binomial model:

$$
\begin{aligned}
C_0 \;=\; & \frac{1}{r_f^3}[p^3 C_{uuu} + 3p^2(1-p)C_{uud} \\
& +3p(1-p)^2 C_{udd} + (1-p)^3 C_{ddd}]
\end{aligned}
$$

- Write down the equilibrium price of a call for an n period binomial model:

$$
\begin{aligned}
C_0 &= \frac{1}{r_f^n} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} C_{u^j d^{(n-j)}} \\
&= \frac{1}{r_f^n} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\
&\quad \max[S_0 u^j d^{(n-j)} - X, 0]
\end{aligned}
$$