REVIEW (BINOMIAL MODEL)

- Define a 1 period binomial model
- Construct a portfolio: buy a number of stocks and invest an amount of money
- Choose the number of stocks and the amount of money in order to replicate the possible payoffs of the put
- Derive the equilibrium price of the put
- Examine the factors that affecting the equilibrium price of the put

- Define a 2 period binomial model

- Derive the equilibrium price of the put for this model

- Derive the equilibrium price of a put for a 3 period binomial model

- Write down the equilibrium price of a put for an n period binomial model

BINOMIAL OPTION PRICING MODEL

Assumptions and Notation

The price of the stock at period zero is denoted by S_0

There are only two periods: 0 and 1.

The are only two possibilities.

The price of the stock in period one can either go up to uS_0, $u > 1$, or down to dS_0, $d < 1$

The price of a put option at period zero is denoted by P_0

The put option expires at period one. There are only two possible states

The two payoffs of the put option are denoted by P_u (in the up state) and P_d (in the down state):

$$P_u = \max(X - uS_0, 0),\ P_d = \max(X - dS_0, 0)$$

We have a risk free security. Its return is denoted by R_f

We assume that

$$d < r_f = R_f + 1 < u$$

METHODOLOGY

PORTFOLIO

1. Form a portfolio. In particular, i) buy an amount H_0 of stocks S_0, and ii) invest an amount of money I_0

The value of this portfolio in period zero is denoted by V_0

In period one the payoff from this portfolio is

$$\begin{aligned} \text{up state} &: \quad H_0 u S_0 + r_f I_0 \\ \text{down state} &: \quad H_0 d S_0 + r_f I_0 \end{aligned}$$

REPLICATE PUT'S PAYOFFS

2. We can choose H_0 and I_0 in such a way in order to replicate the two put's outflows at the end of period 1:

$$\begin{aligned} \text{up state} &: \quad H_0 u S_0 + r_f I_0 = P_u, \qquad (1) \\ \text{down state} &: \quad H_0 d S_0 + r_f I_0 = P_d \qquad (2) \end{aligned}$$

From equation (1) we subtract equation (2):

$$\begin{aligned} H_0 u S_0 - H_0 d S_0 &= P_u - P_d \Rightarrow \\ H_0 S_0 (u - d) &= P_u - P_d \Rightarrow \\ H_0 &= \frac{P_u - P_d}{S_0 (u - d)} \qquad (3) \end{aligned}$$

Thus in equilibrium H_0 is equal to the following ratio: spread in the payoffs of the put over the spread in the price of the stock (in the up and down state)

In other words it is the ratio of the range in possible put values to stock values, often referred to as the hedge ratio or delta value

$$H_0 = \frac{P_u - P_d}{S_0(u-d)}$$

Next, substituting equation (3) into either (1) or (2) gives

$$\begin{aligned} r_f I_0 &= P_u - H_0 u S_0 = P_u - u S_0 \frac{P_u - P_d}{S_0(u-d)} \\ &= \frac{P_u(u-d) - u(P_u - P_d)}{u-d} = \frac{P_d u - P_u d}{u-d} \\ &\Rightarrow I_0 = \frac{P_d u - P_u d}{r_f(u-d)} \end{aligned} \tag{4}$$

Note that since $P_d > P_u$, H_0 will be negative and I_0 will be positive.

This implies that the replicating put portfolio is constructed with a short position in the stock (selling H_0 shares short) and a long position in the risk-free security.

PRICE OF PUT=VALUE OF PORTFOLIO IN PERIOD 0

$$H_0 = \frac{P_u - P_d}{S_0(u-d)}, I_0 = \frac{P_d u - P_u d}{r_f(u-d)}$$

3. Finally by the law of one price you can determine the equilibrium price of the put, P_0, by setting the current put value equal to the current value of the replicating portfolio

That is (using the above equation)

$$\begin{aligned} P_0 &= V_0 = H_0 S_0 + I_0 \\ &= S_0 \frac{P_u - P_d}{S_0(u-d)} + \frac{P_d u - P_u d}{r_f(u-d)} \\ &= \frac{1}{r_f(u-d)}[r_f(P_u - P_d) + P_d u - P_u d] \\ &= \frac{1}{r_f(u-d)}[P_u(r_f - d) + P_d(u - r_f)] \end{aligned}$$

$$P_0 = \frac{1}{r_f(u-d)}[P_u(r_f - d) + CP_d(u - r_f)]$$

Next we can denote

$$\frac{r_f - d}{u - d} = p < 1, \frac{u - r_f}{u - d} = 1 - p < 1,$$

and get

$$P_0 = \frac{1}{r_f}[P_u p + P_d(1 - p)]$$

In other words, the equilibrium price of the put in period 0 is equal to the present value (we divide by $r_f = 1 + R_f$)

of a weighted average of the two possible payoffs of the put in period 1: P_u and P_d

The weights are given by p and $1 - p$.

SPECIAL CASE

As an example examine the special case where

$$\begin{aligned} uS_0 &> X \Rightarrow P_u = 0, \\ dS_0 &< X \Rightarrow P_d = X - dS_0 \end{aligned}$$

If this is the case then

$$P_0 = \frac{1}{r_f} P_d (1 - p) = \frac{(X - dS_0)}{r_f} \left(\frac{u - r_f}{u - d} \right)$$

FACTORS AFFECTING THE PRICE OF PUT

$$P_0 = \frac{(X - dS_0)}{r_f}\left(\frac{u - r_f}{u - d}\right) \tag{5}$$

From the above equation it follows that

i) As $X \uparrow \rightarrow P_0 \uparrow$

ii) As $S_0 \uparrow \rightarrow P_0 \downarrow$

iii) As $r_f \uparrow \longrightarrow P_0 \uparrow\downarrow$ since

$$P_0 = S_0 \frac{P_u - P_d}{S_0(u-d)} + \frac{P_d u - P_u d}{r_f(u-d)} = \frac{dS_0 - X}{(u-d)} + \frac{(X - dS_0)u}{r_f(u-d)}$$

STOCK VOLATILITY

The variance of the stock (σ^2) is given by

$$\sigma^2 = \mathrm{Pr}(u)[uS_0 - E(S)]^2 + \mathrm{Pr}(d)[dS_0 - E(S)]^2$$

where $\mathrm{Pr}(u)$ denotes the probability of being in the up state and $E(S)$ is the expected value of the stock in period 1

The former is 0.5 and the latter is given by

$$\begin{aligned} E(S) &= \mathrm{Pr}(u)uS_0 + \mathrm{Pr}(d)dS_0 \\ &= \frac{S_0}{2}(u+d) \end{aligned}$$

Thus

$$\begin{aligned}\sigma^2 &= \frac{1}{2}[uS_0 - \frac{S_0}{2}(u+d)]^2 + \frac{1}{2}[dS_0 - \frac{S_0}{2}(u+d)]^2 \\ &= S_0^2(\frac{u-d}{2})^2\end{aligned}$$

Thus as either $u \uparrow$ or $d \downarrow \rightarrow \sigma^2 \uparrow$

Finally, note that from equation (5): $P_0 = \frac{(X-dS_0)}{r_f}(\frac{u-r_f}{u-d})$

it follows that as $u \uparrow \longrightarrow P_0 \uparrow$ by $X - dS_0$ (numerator) and $u \uparrow \longrightarrow P_0 \downarrow$ by r_f (denominator). But since $X - dS_0 > r_f$, the overall effect is positive

2 PERIOD BINOMIAL MODEL

Next we assume that we have two periods. In period two there are three possible values for the price of the stock: $u^2S_0 = S_{uu}$, $udS_0 = S_{du}$ and $d^2S_0 = S_{dd}$

Periods		
0	1	2
		$u^2S_0 = S_{uu}$
S_0	$uS_0 = S_u$	$udS_0 = S_{du}$
	$dS_0 = S_d$	$d^2S_0 = S_{dd}$

Accordingly, in the second period there are also three possible payoffs for the put: $P_{uu} = \max(X - S_{uu}, 0)$, $P_{ud} = \max(X - S_{ud}, 0)$ and $P_{dd} = \max(X - S_{dd}, 0)$

Periods		
0	1	2
	P_u	$P_{uu} = \max(X - S_{uu}, 0)$
P_0		$P_{ud} = \max(X - S_{ud}, 0)$
	P_d	$P_{dd} = \max(X - S_{dd}, 0)$

1	2
$P_u = \frac{1}{r_f}[P_{uu}p + P_{ud}(1-p)]$	$P_{uu} = \max(X - S_{uu}, 0)$
$P_d = \frac{1}{r_f}[P_{ud}p + P_{dd}(1-p)]$	$P_{ud} = \max(X - S_{ud}, 0)$
	$P_{dd} = \max(X - S_{dd}, 0)$

When the stock price is at the up stage in period 1 we can use the 1 period Binomial model to price the option in period 1 in the up state:

$$P_u = \frac{1}{r_f}[P_{uu}p + P_{ud}(1-p)] \qquad (6)$$

Similarly when the stock price is at the down stage in period 1 we can use the 1 period Binomial model to price the option in period 1 in the down state:

$$P_d = \frac{1}{r_f}[P_{ud}p + P_{dd}(1-p)] \qquad (7)$$

0	1
$P_0 = \frac{1}{r_f}[P_u p + P_d(1-p)]$	$P_u = \frac{1}{r_f}[P_{uu} p + P_{ud}(1-p)]$ $P_d = \frac{1}{r_f}[P_{ud} p + P_{dd}(1-p)]$

Finally, given the possible put values (P_u and P_d) for period 1 we move to the present and again use the 1 period Binomial model to find P_0:

$$P_0 = \frac{1}{r_f}[P_u p + P_d(1-p)] \tag{8}$$

Substituting equations (6) and (7) into (8) gives:

$$P_0 = \frac{1}{r_f^2}[P_{uu} p^2 + 2p(1-p)P_{ud} + (1-p)^2 P_{dd}]$$

That is P_0 is equal to the present value of a weighted average of the three possible payoffs in period two:

P_{uu}, P_{ud} and P_{dd}.

The sum of the weights is equal to one:

$p^2 + 2p(1-p) + (1-p)^2 = [p + (1-p)]^2 = 1$

3 PERIODS BINOMIAL MODEL

In period 3 there are four possible payoffs for the put:

$P_{uuu} = \max(X - S_{uuu}, 0)$; $P_{uud} = \max(X - S_{uud}, 0)$;

$P_{udd} = \max(X - S_{udd}, 0)$; $P_{ddd} = \max(X - S_{ddd}, 0)$

Periods			
0	1	2	3
			$P_{uuu} = \max(X - S_{uuu}, 0)$
	P_u	P_{uu}	$P_{uud} = \max(X - S_{uud}, 0)$
P_0	P_d	P_{ud}	$P_{udd} = \max(X - S_{udd}, 0)$
		P_{dd}	$P_{ddd} = \max(X - S_{ddd}, 0)$

Steps:

i) We calculate the 3 possible put payoffs in period 2 using the 1 period Binomial model:

$$P_{uu} = \frac{1}{r_f}[P_{uuu}p + P_{uud}(1-p)];$$

$$P_{ud} = \frac{1}{r_f}[P_{uud}p + P_{udd}(1-p)];$$

$$P_{dd} = \frac{1}{r_f}[P_{udd}p + P_{ddd}(1-p)]$$

0	1	2
		P_{uu}
	P_u	
P_0		P_{ud}
	P_d	
		P_{dd}

ii) We calculate the 2 possible put payoffs in period 1 using the 1 period Binomial model:

$$P_u = \frac{1}{r_f}[P_{uu}p + P_{ud}(1-p)];$$

$$P_d = \frac{1}{r_f}[P_{ud}p + P_{dd}(1-p)]$$

iii) We calculate the price of the put in period 0 using the 1 period Binomial model:

$$P_0 = \frac{1}{r_f}[P_u p + P_d(1-p)]$$

Finally, substituting the expressions for P_u, P_d, P_{uu}, P_{ud} and P_{dd} into the expression for P_0 gives

$$
\begin{aligned}
P_0 &= \frac{1}{r_f^3}[p^3_{uuu}P + 3p^2(1-p)P_{uud} \\
&\quad +3p(1-p)^2_{udd}P + (1-p)^3 P_{ddd}]
\end{aligned}
$$

That is the equilibrium price of the put in period 0 is equal to the present value (we divide by r_f^3) of the weighted average of all the 4 possible put payoffs in period 3: $P_{uuu}, P_{uud}, P_{udd}, P_{ddd}$

where the weights can be obtained using the Binomial formula: $\binom{n}{j}p^j(1-p)^{n-j}$ where j is the number of the up states and the $\binom{n}{j} = \frac{n!}{j!(n-j)!}$

n PERIOD BINOMIAL MODEL

In general if we have an n period model the equilibrium price of the put will be given by

$$
\begin{aligned}
P_0 &= \frac{1}{r_f^n} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} P_{u^j d^{(n-j)}} \\
&= \frac{1}{r_f^n} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\
&\quad \max[X - S_0 u^j d^{(n-j)}, 0]
\end{aligned}
$$

If $n = 3$ (we have a 3 period Binomial model) the form will give us

$$
\begin{aligned}
P_0 &= \frac{1}{r_f^3}[p^3 P_{uuu} + 3p^2(1-p)P_{uud} \\
&\quad + 3p(1-p)^2 P_{udd} + (1-p)^3 P_{ddd}]
\end{aligned}
$$

since: $\binom{3}{0} = \frac{3!}{0!3!} = 1$, $\binom{3}{1} = \frac{3!}{2!1!} = 3$, $\binom{3}{2} = \frac{3!}{2!1!} = 3$, $\binom{3}{3} = \frac{3!}{3!0!} = 1$.

SUMMARY (BINOMIAL MODEL)

- Defined a 1 period binomial model:

Periods			Periods	
0	1		0	1
S_0	$uS_0 = S_u$	,	P_0	P_u
	$dS_0 = S_d$			P_d

- Constructed a portfolio: buy a number of stocks and invest an amount of money:

$$\begin{array}{ll} \text{up state:} & H_0 u S_0 + r_f I_0 \\ \text{down state} \ : & H_0 d S_0 + r_f I_0 \end{array}$$

- Chosen the number of stocks and the amount of money in order to replicate the possible payoffs of the put:

$$H_0 = \frac{P_u - P_d}{S_0(u-d)}, I_0 = \frac{P_d u - P_u d}{r_f(u-d)}$$

- Derived the equilibrium price of the put:

$$P_0 = \frac{1}{r_f}[P_u p + P_d(1-p)],$$

$$\frac{r_f - d}{u-d} = p < 1, \frac{u - r_f}{u-d} = 1 - p < 1,$$

- Examined the factors that affecting the equilibrium price of the put:

i) As $X \uparrow \rightarrow P_0 \uparrow$; ii) As $S_0 \uparrow \rightarrow P_0 \downarrow$; iii) As $r_f \uparrow \longrightarrow P_0 \downarrow$; As $\sigma^2 \uparrow \longrightarrow P_0 \uparrow$

- Defined a 2 period binomial model:

Periods

0	1	2
		$P_{uu} = \max(X - S_{uu}, 0)$
	P_u	
P_0		$P_{ud} = \max(X - S_{ud}, 0)$
	P_d	
		$P_{dd} = \max(X - S_{dd}, 0)$

- Derived the equilibrium price of the put

$$P_0 = \frac{1}{r_f^2}[P_{uu}p^2 + 2p(1-p)P_{ud} + (1-p)^2 P_{dd}]$$

- Derive the equilibrium price of a put for an 3 period binomial model:

$$\begin{aligned} P_0 &= \frac{1}{r_f^3}[p^3 P_{uuu} + 3p^2(1-p)P_{uud} \\ &\quad +3p(1-p)^2 P_{udd} + (1-p)^3 P_{ddd}] \end{aligned}$$

- Write down the equilibrium price of a put for an n period binomial model:

$$
\begin{aligned}
P_0 &= \frac{1}{r_f^n} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} P_{u^j d^{(n-j)}} \\
&= \frac{1}{r_f^n} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\
&\quad \max[X - S_0 u^j d^{(n-j)}, 0]
\end{aligned}
$$