Problem Set 3

Question 1. Solve exercise 2.4 in Johnston and Dinardo:

Prove that:

$$\frac{dY}{dX}\frac{X}{Y} = \frac{d\,\mathsf{ln}(Y)}{d\,\mathsf{ln}(X)}.$$

Answer. We will make use of the following

$$\begin{aligned} \frac{d\,\mathsf{ln}(Y)}{dY} &= \frac{1}{Y} \Rightarrow d\,\mathsf{ln}(Y) = \frac{dY}{Y}, && (1)\\ \frac{d\,\mathsf{ln}(X)}{dX} &= \frac{1}{X} \Rightarrow d\,\mathsf{ln}(X) = \frac{dX}{X}, && (2) \end{aligned}$$

Dividing equation (1) by equation (2) gives

$$\frac{d\,\mathsf{ln}(Y)}{d\,\mathsf{ln}(X)} = \frac{\frac{dY}{Y}}{\frac{dX}{X}} = \frac{dY}{dX}\frac{X}{Y} = e^{Y}_{X}.$$

2. Solve exercise 2.5 in Johnston and Dinardo:

A response rate to a stimulus X is modeled by the function

$$\frac{100}{100 - Y} = a + \frac{b}{X},$$

where Y is measured in percentage terms. Outline the properties of this function and sketch its graph. Fit the function to the accompanying data.

X	3	7	12	17	25	35	45	55	70	120
Y	86	79	76	69	65	62	52	51	51	48

Answer:

$$\frac{100}{100 - Y} = a + \frac{b}{X},$$

To sketch the graph we need to find the two asymptotes:

When $X \rightarrow \infty$, $Y \rightarrow 100(1 - \frac{1}{a})$. When $Y \rightarrow \infty$, $X \rightarrow -\frac{b}{a}$.

In addition, when $X \rightarrow 0$, $Y \rightarrow 100$

$$\frac{100}{100 - Y} = a + \frac{b}{X},$$

To fit the function in the accompanying data we use the following transformation:

$$\begin{aligned} Y^* &= \frac{100}{100 - Y}, \\ X^* &= \frac{1}{X}. \end{aligned}$$

The two estimated values are given by:

$$\begin{aligned} \beta &= \frac{\sum y^* x^*}{\sum (x^*)^2} = \cdots , \\ \alpha &= \overline{Y^*} - \beta \overline{X^*} = \cdots . \end{aligned}$$

where $y^* = Y^* - \overline{Y}^*$ and $x^* = X^* - \overline{X}^*$

3. Solve exercise 2.8 in Johnston and Dinardo:

A theorist postulates that the following functional form will provide a good fit to a set of data on Y and X:

$$Y = a + b\frac{1}{1 - X}.$$

Sketch the graph of this function when a and b are both positive. Three sample observations give these values:

Y	0	5	6
X	1.2	1.5	2.0

Fit the forgoing function to the data. If Y denotes the per capita consumption of peanuts and X denotes income, give a point estimate of the peanut consumption of a millionaire.

Answer: To sketch the graph we need to find the two asymptotes:

When $X \to \infty$, $Y \to a$. When $Y \to \infty$, $X \to 1$.

To fit the function in the accompanying data we use the following transformation:

$$X^* = \frac{1}{1 - X}.$$

thus

X	1.2	1.5	2.0	
X^*	-5	-2	-1	$\overline{X}^* = -2.67$
x^*	-2.33	0.67	1.67	
$(x^*)^2$	5.43	0.45	2.79	$\sum (x^*)^2 = 8.67$
Y	0	5	6	$\overline{Y} = 3.67$
y	-3.67	1.33	2.33	
x^*y	8.55	0.89	3.89	$\sum x^*y = 13.33.$

Next we estimate the following regression:

$$Y = a + bX^* + e.$$

Recall that $\sum x^* y = 13.33$, $\sum (x^*)^2 = 8.67$, and that $\overline{Y} = 3.67$; $\overline{X}^* = -2.67$

The estimated values of a and b are given by

$$\begin{aligned} \beta &= \frac{\sum yx^*}{\sum (x^*)^2} = \frac{13.33}{8.67} = 1.54, \\ \alpha &= \overline{Y} - \beta \overline{X}^* = 3.67 + 1.54 \times 2.67 = 7.78. \end{aligned}$$

The peanut consumption of a millionaire is $\alpha = 7.78$.

4. Solve exercise 2.9 in Johnston and Dinardo:

An economist hypothesizes that the average production cost of an article declines with increasing batch size, tending toward an asymptotic minimum value. Some sample data from the process are as follows:

Batch size	1	5	10	20
Average cost	31	14	12	11

Fit a curve of the form

$$Y = a + b\frac{1}{X},$$

to these data, where Y denotes average cost and X indicates batch size. What is the estimated minimum cost level? Estimate the batch size that would be required to get the average cost to within 10 percent of this minimum level.

Answer: To fit the function in the accompanying data we use the following transformation:

$$X^* = \frac{1}{X}.$$

Thus

X	1	5	10	20	
X^*	1	0.2	0.10	0.05	$\overline{X}^* = 0.34$
x^*	0.66	-0.14	-0.24	-0.29	
$(x^*)^2$	0.43	0.02	0.06	0.08	$\sum (x^*)^2 = 0.59$
Y	31	14	12	11	$\overline{Y} = 17$
y	14	-3	-5	-6	
yx^*	9.24	0.42	1.20	1.74	$\sum yx^* = 12.60$

The estimated values of a and b are given by

$$\begin{aligned}\beta &= \frac{\sum yx^*}{\sum (x^*)^2} = \frac{12.60}{0.59} = 21.35, \\ \alpha &= \overline{Y} - \beta \overline{X}^* = 17 - 21.35 \times 0.34 = 9.74\end{aligned}$$

The estimated minimum cost level is α.

We also want $Y = 1.1\alpha$. Thus

$$\begin{aligned} 1.1\alpha &= \alpha + \beta\frac{1}{X} \Rightarrow 0.1\alpha = \frac{\beta}{X} \Rightarrow \\ X &= \frac{\beta}{0.1\alpha} = \frac{10\beta}{\alpha} = \frac{213.5}{9.74} = 21.92. \end{aligned}$$

5. Exercise 1.16 in Johnston and Dinardo

A sample of 20 observations corresponding to the model

$$Y = a + bX + e,$$

where the e's are normally and independently distributed with zero mean and constant variance, gave the following data:

$$\begin{aligned}
\sum Y &= 21.9, \ \sum(Y - \overline{Y})^2 = 86.9, \\
\sum(X - \overline{X})(Y - \overline{Y}) &= 106.4, \\
\sum X &= 186.2, \ \sum(X - \overline{X})^2 = 215.4.
\end{aligned}$$

Calculate the standard errors of the estimates of a and b.

Answer: The forms for the estimated standard errors are:

$$\begin{aligned}
\text{Var}(\beta) &= \frac{s^2}{\sum x^2}, \\
\text{Var}(\alpha) &= s^2[\frac{1}{N} + \frac{\overline{X}^2}{\sum x^2}].
\end{aligned}$$

The s^2 is given by

$$s^2 = \frac{\text{RSS}}{N-2} = \frac{\sum \hat{e}^2}{N-2}.$$

Moreover, we have

$$\begin{aligned}
\text{RSS} &= \sum y^2 - r^2 \sum y^2 = \\
&= \sum y^2 - \frac{(\sum yx)^2}{\sum x^2 \sum y^2} \sum y^2 = \\
&= \sum y^2 - \frac{(\sum yx)^2}{\sum x^2} = 86.9 - \frac{106.4^2}{215.4} = 34.34.
\end{aligned}$$

Thus

$$s^2 = \frac{34.34}{18} = 1.91.$$

Finally, we have

$$\begin{aligned}
\text{Var}(\beta) &= \frac{s^2}{\sum x^2} = \frac{1.91}{215.4} = 0.009, \\
\text{Var}(\alpha) &= s^2[\frac{1}{N} + \frac{\overline{X}^2}{\sum x^2}] = \\
&= 1.91[\frac{1}{20} + \frac{9.31^2}{215.4}] = 0.86
\end{aligned}$$

(Not this one) 6. Explain why the effect of trading volume on stock volatility might be negative.

Answer: See notes: Volume-Volatility link.