BRUNEL UNIVERSITY
Master of Science Degree examination

Test Exam Paper 2005-2006

EC5002: Modelling Financial Decisions and Markets
EC5030: Introduction to Quantitative methods

Model Answers

1. COMPULSORY

(a)

$$\beta = \frac{\sum xy}{\sum x^2} = \frac{70}{40} = 1.75$$

(b) The r^2 from the regression of y on x is the variation in y that is explained (linearly) by x.

$$r^2 = \frac{\text{ESS}}{\text{TSS}} = \frac{\text{TSS-RSS}}{\text{TSS}} = 1 - \frac{\text{RSS}}{\text{TSS}}.$$

(c) i) Inflation (INFL) is regressed on the unemployment rate (UNR):

$$\text{INFL}{=}a + b\text{UNR} + u.$$

ii) A lagged response is not unreasonable since time is required for unemployment to affect wages and further time for wage changes to filter through to the prices of goods:

$$\text{INFL}{=}a + b\text{UNR(-1)} + u.$$

iii) One can fit the reciprocal relation:

$$\text{INFL}{=}a + b\frac{1}{\text{UNR(-1)}} + u.$$

iv) Finally, one can fit (by nonlinear least squares) the nonlinear relation:

$$\text{INFL}{=}a + b\frac{1}{\text{UNR(-1)-}\gamma} + u.$$

(d) The specification of the linear model centers on the disturbance vector u and the X matrix. The assumptions are as follows:

$$y = X\beta + u,$$

with

$$\text{E}(u) = 0, \ \text{E}(uu') = \sigma^2 I.$$

In other words, u_i, $i = 1, \ldots, n$ are *iid* $N(0, \sigma^2)$. Moreover, X is nonstochastic with full column rank k.

(e) The Box-Pierce statistic is based on the squares of the first p autocorrelation coefficients of the OLS residuals. The statistic is defined as

$$Q = n\sum_{j=1}^{p} r_j^2,$$

where

$$r_j = \frac{\sum_{t=j+1}^{n} e_t e_{t-j}}{\sum_{t=1}^{n} e_t^2}.$$

Under the hypothesis of zero autocorrelations for the residuals, Q will have an asymptotic χ^2 distribution, with degrees of freedom equal to p minus the number of parameters estimated in fitting the ARMA model. An improved small-sample performance is expected from the revised Ljung-Box statistic

$$Q' = n(n+2)\sum_{j=1}^{p} r_j^2/(n-j).$$

(f) The hypothesis is $\beta_k + \beta_l = 1$ with test statistic

$$\begin{aligned} t &= \frac{b_k + b_l - 1}{\sqrt{\mathrm{var}(b_k + b_l)}} = \\ &= \frac{b_k + b_l - 1}{\sqrt{\mathrm{var}(b_k) + \mathrm{var}(b_l) + 2\mathrm{cov}(b_k, b_l)}} \\ &= \frac{0.632 + 0.452 - 1}{\sqrt{0.257^2 + 0.219^2 + 2 \times 0.055}} = 0.178. \end{aligned}$$

In order to get the critical values we need the sample size. This is not significant for any sample size. We cannot reject the hypothesis of constant returns to scale.

2. (a) See Lecture Notes.

(b) In all cases the null hypothesis is rejected. Therefore, there is evidence of: heteroscedasticity; serial correlation in the residuals and squared residuals; Non-normality.

Finally, according to Ramsey's reset test the null hypothesis: $H_0 : u \sim N(0, \sigma^2 I)$ is rejected.

3. (a) If the unknown vector β is replaced by some guess or estimate b, this defines a vector of residuals e,

$$\underset{(n\times 1)}{e} = \underset{(n\times 1)}{Y} - \underset{(n\times k)}{X}\underset{(k\times 1)}{b},$$

The least-squares principle is to choose b to minimize the residual sum of squares, $\sum_{i=1}^{n} e_i^2 = e'e$; $e' = [e_1 \ e_2 \ \cdots e_n]$:

$$\begin{aligned} e'e &= (Y - Xb)'(Y - Xb) \\ &= (Y' - b'X')(Y - Xb) \\ &= Y'Y - b'X'Y - Y'Xb + b'X'Xb. \end{aligned}$$

Next, note that $\underset{(1\times n)}{Y'}\underset{(n\times k)}{X}\underset{(k\times 1)}{b}$ is a scalar and hence its transpose $(Y'Xb)' = b'X'Y$ is the same scalar $(Y'Xb)$. Thus, $b'X'Y = Y'Xb$.

From the above analysis it follows that

$$e'e = Y'Y - 2b'X'Y + b'X'Xb.$$

The first order conditions are

$$\frac{\partial(e'e)}{\partial b} = -2X'Y + 2X'Xb = 0,$$

giving the least-squares b vector as a function of the data:

$$\begin{aligned} X'Xb &= X'Y, \text{ or} \\ b &= (X'X)^{-1}X'Y. \end{aligned}$$

(b) Under rational expectations, the regression

$$r = \alpha + \beta r^* + u,$$

should yield $a = 0$ and $b = 1$. We can carry out this test by the F-test in part a. In this case:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The test statistic is

$$\begin{aligned} F &= \frac{(b-r)'(X'X)(b-r)/q}{s^2} = \\ &= \frac{\begin{bmatrix} 0.24 & -0.06 \end{bmatrix} \begin{bmatrix} 30 & 300 \\ 300 & 3052 \end{bmatrix} \begin{bmatrix} 0.24 \\ -0.06 \end{bmatrix} /2}{28.56/28} = 1.998. \end{aligned}$$

The 5% critical value is 3.34. The test statistic is not significant and we cannot reject the rational expectations hypothesis.

4. (a) i)

$$\underset{(q\times k)}{R} \times \underset{(k\times 1)}{b} = \underset{(q\times 1)}{r}.$$

Assume that $k = 3$. Then

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Example 1: The null hypothesis is: $H_0 : b_2 = b_3$ or $b_2 - b_3 = 0$ $(q = 1)$.

Example 2: The null hypothesis is: $H_0 : b_2 = b_3 = 0$ $(q = 2)$.

For the foregoing examples we have:

$$\begin{aligned} 1)\ R &= \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, r = 0, \\ 2)\ R &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

ii) The vector of the restrictions is: Rb-r.

The estimated variance covariance matrix of the coefficient vector (b) is: $\text{var}(b) = s^2(X'X)^{-1}$. Thus, $\text{var}(Rb) = \text{var}(Rb - r) = s^2R(X'X)^{-1}R'$. Note that $(X'X)^{-1}$ is a $(k \times k)$ matrix, R is a $(q \times k)$ matrix and R' is a $(k \times q)$ matrix. In other words $var(Rb - r)$ is a $(q \times q)$ matrix.

A computable statistic, which has an F-distribution under the null hypothesis is

$$(Rb - r)'[\text{var}(Rb - r)]^{-1}(Rb - r)/q \sim F(q, n - k).$$

or

$$(Rb - r)'[s^2R(X'X)^{-1}R']^{-1}(Rb - r)/q \sim F(q, n - k).$$

Note that $(Rb - r)$ is a $(q \times 1)$ column vector, $(Rb - r)'$ is a $(1 \times q)$ row vector and $[s^2R(X'X)^{-1}R']^{-1}$ is a $(q \times q)$ matrix. In other words the computable statistic is a scalar.

iii) The test procedure is then to reject the null hypothesis $Rb = r$ if the computed F value exceeds a preselected critical value.

(b) The likelihood function is given by

$$\begin{aligned} L(\theta; x) &= \prod_{i=1}^{n} f(x_i) = \\ &= \prod_{i=1}^{n} \theta e^{-\theta x_i} \\ &= \theta^n e^{-\theta \sum_{i=1}^{n} x_i} \end{aligned}$$

Therefore, the log-likelihood function is

$$l = \ln L = n\ln\theta - \theta \sum_{i=1}^{n} x_i.$$

The first order condition is

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0,$$

which gives the MLE

$$\widehat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\overline{x}}.$$

(c) In the two-variable equation we have:

$$\begin{aligned} \text{Var}(b) &= \begin{bmatrix} \text{Var}(a) & \text{Cov}(a, b) \\ \text{Cov}(a, b) & \text{Var}(b) \end{bmatrix} \\ &= \sigma^2(X'X)^{-1}. \end{aligned}$$

$X'X$ in this case is

$$X'X = \begin{bmatrix} N & \sum X \\ \sum X & \sum X^2 \end{bmatrix}.$$

Thus,

$$(X'X)^{-1} = \frac{\begin{bmatrix} \sum X^2 & -\sum X \\ -\sum X & N \end{bmatrix}}{N \sum X^2 - (\sum X)^2}.$$

Then, we have

$$\begin{aligned}
\mathrm{Var}(b) &= \sigma^2 \frac{N}{N\sum X^2 - (\sum X)^2} = \\
& \sigma^2 \frac{1}{\sum X^2 - (\sum X)^2/N} \\
&= \sigma^2 \frac{1}{\sum X^2 - N\overline{X}^2} \\
&= \sigma^2 \frac{1}{\sum (X-\overline{X})^2}.
\end{aligned}$$

5. (a) One simply computes an auxiliary regression of the squared OLS residuals on a constant and all nonredundant variables in the set consisting of the regressors, their squares, and their cross products. Suppose, for example, that

$$\widehat{y}_t = a + b_1 x_{1t} + b_2 x_{2t},$$

and $e_t = y_t - \widehat{y}_t$. Then, the auxiliary regression is e_t^2 on $1, x_{1t}, x_{2t}, x_{1t}^2, x_{2t}^2$, and $x_{1t}x_{2t}$. On the hypothesis of homoscedasticity, nR^2 is asymptotically distributed as $\chi^2(5)$. The degrees of freedom are the number of variables in the auxiliary regression (excluding the constant). In general, under the null of homoscedasticity,

$$nR^2 \overset{a}{\sim} \chi^2(q),$$

where q is the number of variables in the auxiliary regression less than one. If homoscedasticty is rejected, there is no indication of the form of the heteroscedasticity and, thus no guide to an appropriate estimator. Computing the White standard errors would be, however, wise if one proceeds with OLS.

(b) Recall that the least-squares vector b is related to the data

$$b = (X'X)^{-1}X'Y,$$

where

$$Y = X\beta + u.$$

Substituting for Y gives

$$\begin{aligned}
b &= (X'X)^{-1}X'(X\beta + u) \\
&= \beta + (X'X)^{-1}X'u. \qquad (1)
\end{aligned}$$

In taking expectations we have $\mathrm{E}(b) = \beta$.

Conventional OLS coefficient standard errors are incorrect, and the conventional test statistics based on them are invalid. The correct variance matrix for the OLS coefficient is

$$\begin{aligned}
\mathrm{Var}(b) &= \mathrm{E}[(X'X)^{-1}X'uu'X(X'X)^{-1}] \\
&= (X'X)^{-1}X'\mathrm{E}(uu')X(X'X)^{-1} \\
&= (X'X)^{-1}X'\Omega X(X'X)^{-1}. \qquad (2)
\end{aligned}$$

The conventional formula calculates $\sigma^2(X'X)^{-1}$, which is only part of the correct expression in (2).

The White estimator replaces the unknown σ_t^2 $(t = 1, \ldots, n)$ by e_t^2, where the e_t denote the OLS residuals. This provides a consistent estimator of the variance matrix for the OLS coefficient vector and is particularly useful because it does not require any specific assumptions about the form of the heteroscedasticity. The empirical implementation of Eq. (2) is then

$$\text{Estimated Var}(b) = (X'X)^{-1}X'\widehat{\Omega}X(X'X)^{-1},$$

where

$$\widehat{\Omega} = \text{diag}\{e_1^2, \ldots, e_n^2\}.$$

The square roots of the elements on the principal diagonal of estimated Var(b) are the estimated standard errors of the OLS coefficients. They are often referred to as heteroscedasticity-consistent standard errors.

(c) Ramsey has argued that the various sepicification errors (omitted variables, incorrect functional form, correlation between X and u) give rise to a nonzero u vector. Thus the null and alternative hypotheses are

$$\begin{aligned} H_0 &: u \sim N(0, \sigma^2 I), \\ H_1 &: u \sim N(\mu, \sigma^2 I). \end{aligned}$$

The test of H_0 is based on the augmented regression

$$y = X\beta + Za + u.$$

The test for the specification error is then $a = 0$. ramsey's suggestion is that Z shold contain powers of the predicted values of the dependent variable. For example:

$$Z = [\widehat{y}^2 \; \widehat{y}^3 \; \widehat{y}^4]$$

where $\widehat{y} = Xb$, and $\widehat{y}^2 = [\widehat{y}_1^2 \; \widehat{y}_2^2 \ldots \widehat{y}_n^2]$.