BRUNEL UNIVERSITY
Master of Science Degree examination

Specimen Exam Paper 2005-2006

EC5002: Modelling Financial decisions and Markets
EC5030: Introduction to Quantitative methods

Model Answers

1. COMPULSORY

(a)

$$\beta = \frac{\sum xy}{\sum x^2} = \frac{106.04}{215.4}$$

(b) The r^2 from the regression of y on x is the variation in y that is explained (linearly) by x. Note that this does not have any causal interpretation.

$$b_{yx}b_{xy} = \frac{\sum xy}{\sum x^2}\frac{\sum xy}{\sum y^2} = r^2$$

(c) i) Inflation (INFL) is regressed on the unemployment rate (UNR):

$$\text{INFL}{=}a + b\text{UNR} + u.$$

ii) A lagged response is not unreasonable since time is required for unemployment to affect wages and further time for wage changes to filter through to the prices of goods:

$$\text{INFL}{=}a + b\text{UNR(-1)} + u.$$

iii) One can fit the reciprocal relation:

$$\text{INFL}{=}a + b\frac{1}{\text{UNR(-1)}} + u.$$

iv) Finally, one can fit (by nonlinear least squares) the nonlinear relation:

$$\text{INFL}{=}a + b\frac{1}{\text{UNR(-1)-}\gamma} + u.$$

(d) Significantly autocorrelated disturbances would be an indication of an inadequate specification. Suppose, for example, that the relationship really is

$$y_t = \beta_1 + \beta_2 x_t + \beta_3 y_{t-1} + u_t,$$

and u_t is white noise. A researcher specifies

$$y_t = \beta_1 + \beta_2 x_t + v_t,$$

The pseudodisturbance is then $v_t = \beta_3 y_{t-1} + u_t$, which is autocorrelated because the correct specification makes y autocorrelated.

(e) Suppose the specified equation is

$$y_t = \beta_1 + \beta_2 x_t + u_t, \tag{1}$$

with

$$u_t = \beta_3 u_{t-1} + v_t,$$

where it is assumed that $|\beta_3| < 1$ and that the v's are independently and identically distributed normal variables with zero mean and variance σ_v^2. We wish to test the hypothesis that $\beta_3 = 0$.

The test is obtained in two steps. First apply OLS to Eq. (1) to obtain the residuals e_t. Then regress e_t on $[1 \ x_t \ e_{t-1}]$ to find R^2. Under H_0 nR^2 is asymptotically $\chi^2(1)$.

(f) The hypothesis is $\beta_k - \beta_l = 0$ with test statistic

$$\begin{aligned} t &= \frac{b_k - b_l}{\sqrt{\text{var}(b_k - b_l)}} = \\ &= \frac{b_k - b_l}{\sqrt{\text{var}(b_k) + \text{var}(b_l) - 2\text{cov}(b_k, b_l)}} \\ &= \frac{0.632 - 0.452}{\sqrt{0.257^2 + 0.219^2 - 2 \times 0.055}} = 2.846. \end{aligned}$$

In order to get the critical values we need the sample size. Looking at the t-table we find that if $n - k > 3$ or $n > 6$, we reject the hypothesis at 5% significance level.

2. (a) See Lecture Notes.

(b) In all cases the null hypothesis is rejected. Therefore, there is evidence of: heteroscedasticity; serial correlation in the squared residuals; Non-normality; ARCH effects.

Finally, according to Ramsey's reset test the null hypothesis: $H_0 : u \sim N(0, \sigma^2 I)$ is rejected.

3. (a) i)

$$\underset{(q\times k)}{R} \times \underset{(k\times 1)}{b} = \underset{(q\times 1)}{r}.$$

Assume that $k = 3$. Then

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Example 1: The null hypothesis is: $H_0 : b_2 = b_3$ or $b_2 - b_3 = 0$ $(q = 1)$.

Example 2: The null hypothesis is: $H_0 : b_2 = b_3 = 0$ $(q = 2)$.

For the foregoing examples we have:

$$\begin{aligned} 1)\ R &= \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, r = 0, \\ 2)\ R &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

ii) The vector of the restrictions is: Rb-r.

The estimated variance covariance matrix of the coefficient vector (b) is: var$(b) = s^2(X'X)^{-1}$. Thus, var(Rb) =var$(Rb - r) = s^2R(X'X)^{-1}R'$. Note that $(X'X)^{-1}$ is a $(k \times k)$ matrix, R is a $(q \times k)$ matrix and R' is a $(k \times q)$ matrix. In other words $var(Rb - r)$ is a $(q \times q)$ matrix.

A computable statistic, which has an F-distribution under the null hypothesis is

$$(Rb - r)'[\text{var}(Rb - r)]^{-1}(Rb - r)/q \sim F(q, n - k).$$

or

$$(Rb - r)'[s^2R(X'X)^{-1}R']^{-1}(Rb - r)/q \sim F(q, n - k).$$

Note that $(Rb - r)$ is a $(q \times 1)$ column vector, $(Rb - r)'$ is a $(1 \times q)$ row vector and $[s^2R(X'X)^{-1}R']^{-1}$ is a $(q \times q)$ matrix. In other words the computable statistic is a scalar.

iii) The test procedure is then to reject the null hypothesis $Rb = r$ if the computed F value exceeds a preselected critical value.

(b) Under rational expectations, the regression

$$r = \alpha + \beta r^* + u,$$

should yield $a = 0$ and $b = 1$. We can carry out this test by the F-test in part a. In this case:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The test statistic is

$$\begin{aligned} F &= \frac{(b - r)'(X'X)(b - r)/q}{s^2} = \\ &= \frac{\begin{bmatrix} 0.24 & -0.06 \end{bmatrix} \begin{bmatrix} 30 & 300 \\ 300 & 3052 \end{bmatrix} \begin{bmatrix} 0.24 \\ -0.06 \end{bmatrix} /2}{28.56/28} = 1.998. \end{aligned}$$

The 5% critical value is 3.34. The test statistic is not significant and we cannot reject the rational expectations hypothesis.

4. The specification of the linear model centers on the disturbance vector u and the X matrix. The assumptions are as follows:

$$y = X\beta + u,$$

with

$$\text{E}(u) = 0, \ \text{E}(uu') = \sigma^2 I.$$

In other words, u_i, $i = 1, \ldots, n$ are *iid* $N(0, \sigma^2)$. Moreover, X is nonstochastic with full column rank k.

(b) Recall that the least-squares vector b is related to the data

$$b = (X'X)^{-1}X'Y,$$

where

$$Y = X\beta + u.$$

Substituting for Y gives

$$\begin{aligned} b &= (X'X)^{-1}X'(X\beta + u) \\ &= \beta + (X'X)^{-1}X'u. \end{aligned} \tag{2}$$

In taking expectations we have $\mathrm{E}(b) = \beta$. The variance-covariance matrix of the LS estimates is established as follows:

$$\mathrm{Var}(b) = \mathrm{E}[(b-\beta)(b-\beta)'].$$

If we substitute from (2),

$$\begin{aligned} \mathrm{Var}(b) &= \mathrm{E}[(X'X)^{-1}X'uu'X(X'X)^{-1}] \\ &= (X'X)^{-1}X'\mathrm{E}(uu')X(X'X)^{-1} \\ &= (X'X)^{-1}X'\sigma^2 IX(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}X'X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}. \end{aligned}$$

(c) In the two-variable equation we have:

$$\begin{aligned} \mathrm{Var}(b) &= \begin{bmatrix} \mathrm{Var}(a) & \mathrm{Cov}(a,b) \\ \mathrm{Cov}(a,b) & \mathrm{Var}(b) \end{bmatrix} \\ &= \sigma^2(X'X)^{-1}. \end{aligned}$$

$X'X$ in this case is

$$X'X = \begin{bmatrix} N & \sum X \\ \sum X & \sum X^2 \end{bmatrix}.$$

Thus,

$$(X'X)^{-1} = \frac{\begin{bmatrix} \sum X^2 & -\sum X \\ -\sum X & N \end{bmatrix}}{N\sum X^2 - (\sum X)^2}.$$

Then, we have

$$\begin{aligned} \mathrm{Cov}(a,b) &= \sigma^2 \frac{-\sum X}{N\sum X^2 - (\sum X)^2} = \\ & \sigma^2 \frac{-\overline{X}}{\sum X^2 - (\sum X)^2/N} \\ &= \sigma^2 \frac{-\overline{X}}{\sum X^2 - N\overline{X}^2} \\ &= \sigma^2 \frac{-\overline{X}}{\sum (X - \overline{X})^2}. \end{aligned}$$

5. (a) One simply computes an auxiliary regression of the squared OLS residuals on a constant and all nonredundant variables in the set consisting of the regressors, their squares, and their cross products. Suppose, for example, that

$$\widehat{y}_t = a + b_1 x_{1t} + b_2 x_{2t},$$

and $e_t = y_t - \widehat{y}_t$. Then, the auxiliary regression is e_t^2 on $1, x_{1t}, x_{2t}, x_{1t}^2, x_{2t}^2$, and $x_{1t}x_{2t}$. On the hypothesis of homoscedasticity, nR^2 is asymptotically distributed as $\chi^2(5)$. The degrees of freedom are the number of variables in the auxiliary regression (excluding the constant). In general, under the null of homoscedasticity,

$$nR^2 \overset{a}{\sim} \chi^2(q),$$

where q is the number of variables in the auxiliary regression less than one. If homoscedasticty is rejected, there is no indication of the form of the heteroscedasticity and, thus no guide to an appropriate estimator. Computing the White standard errors would be, however, wise if one proceeds with OLS.

(b) Recall that the least-squares vector b is related to the data

$$b = (X'X)^{-1}X'Y,$$

where

$$Y = X\beta + u.$$

Substituting for Y gives

$$\begin{aligned} b &= (X'X)^{-1}X'(X\beta + u) \\ &= \beta + (X'X)^{-1}X'u. \end{aligned} \tag{3}$$

In taking expectations we have $\mathrm{E}(b) = \beta$.

Conventional OLS coefficient standard errors are incorrect, and the conventional test statistics based on them are invalid. The correct variance matrix for the OLS coefficient is

$$\begin{aligned} \mathrm{Var}(b) &= \mathrm{E}[(X'X)^{-1}X'uu'X(X'X)^{-1}] \\ &= (X'X)^{-1}X'\mathrm{E}(uu')X(X'X)^{-1} \\ &= (X'X)^{-1}X'\Omega X(X'X)^{-1}. \end{aligned} \tag{4}$$

The conventional formula calculates $\sigma^2(X'X)^{-1}$, which is only part of the correct expression in (4).

The White estimator replaces the unknown σ_t^2 $(t = 1, \ldots, n)$ by e_t^2, where the e_t denote the OLS residuals. This provides a consistent estimator of the variance matrix for the OLS coefficient vector and is particularly useful because it does not require any specific assumptions about the form of the heteroscedasticity. The empirical implementation of Eq. (4) is then

$$\text{Estimated Var}(b) = (X'X)^{-1}X'\widehat{\Omega}X(X'X)^{-1},$$

where

$$\widehat{\Omega} = \mathrm{diag}\{e_1^2, \ldots, e_n^2\}.$$

The square roots of the elements on the principal diagonal of estimated Var(b) are the estimated standard errors of the OLS coefficients. They are often referred to as heteroscedasticity-consistent standard errors.

(c) Testing for ARCH:

i) Fit Y to X by OLS and obtain the residuals $\{e_t\}$.

ii) Compute the OLS regression, $e_t^2 = a_0 + a_1 e_{t-1}^2 + \ldots + a_p e_{t-p}^2$.

iii) Test the joint significance of $a_1, \ldots, a_p$.

If these coefficients are significantly different from zero, the assumption of conditionally homoscedastic disturbances is rejected in favor of ARCH disturbances.