TESTING LINEAR HYPOTHESES ABOUT β

Consider the following regression

$$Y_i = b_1 + b_2 X_{2i} + b_3 X_{3i} + b_4 X_{4i} + e_i$$

We want to test 3 linear restrictions regarding the 4 b's

The restrictions can be written in the following form

$$\underbrace{\begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \end{bmatrix}}_{(3x4)} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{(4x1)} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}}_{(3x1)}$$

or

$$\underset{(3x4)}{R}\ \underset{(4x1)}{b} = \underset{(3x1)}{r}$$

The order of the matrix R is 3x4: 3 (the rows) is the number of restrictions and 4 is the number of coefficients to be estimated

EXAMPLE 1

We want to test the following hypothesis

$$H_0 : b_2 = b_3 = b_4 = 0$$

Notice that we have three restrictions. These can be written as follows

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(3x4)} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{(4x1)} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{(3x1)}$$

Notice that the first equation is the first restriction: $b_2 = 0$

EXAMPLE 2

We want to test the following restrictions:

$$b_2 - b_3 = 0,\ b_4 = 0$$

We can write the two restrictions as follows:

$$\underbrace{\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(2x4)} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{(4x1)} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{(2x1)}$$

Notice that the first equation is the first restriction: $b_2 - b_3 = 0$

EXAMPLE 3

We want to test the hypothesis: $b_2 + b_3 = 1$

If b_2 and b_3 are labor and capital elasticities in a production function, this formulation hypothesizes constant returns to scale

We can write this restriction as

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}}_{(1x4)} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{(4x1)} = \underbrace{[1]}_{(1x1)}$$

GENERAL CASE

In general if we have q restrictions and k coefficients to be estimated we can write

$$\underset{(qxk)}{R}\underset{(kx1)}{b} = \underset{(qx1)}{r}$$

or

$$\underbrace{\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1k} \\ R_{21} & R_{22} & \cdots & R_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ R_{q1} & R_{q2} & \cdots & R_{qk} \end{bmatrix}}_{(qxk)} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}}_{(kx1)} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_q \end{bmatrix}}_{(qx1)}$$

Thus the first restriction is given by

$$R_{11}b_1 + R_{12}b_2 + \cdots + R_{1k}b_k = r_1$$

and the qth one by

$$R_{q1}b_1 + R_{q2}b_2 + \cdots + R_{qk}b_k = r_q$$

Under the null

$$\underset{(qx1)}{(R\beta - r)} \sim N[0, \sigma^2 R\underset{(qxq)}{(X'X)^{-1}}R']$$

$R\beta - r$ are the estimated restrictions

F DISTRIBUTION

It can be shown that

$$\frac{\underset{(1xq)}{(R\beta - r)'}\underset{(qxq)}{[s^2 R(X'X)^{-1}R']^{-1}}\underset{(qx1)}{(R\beta - r)}}{q} \sim F(q, N-k)$$

EXAMPLE 1

Assume that we have three coefficients to estimate: b_1, b_2, b_3 and we want to test the null hypothesis that $b_2 = 0$

Since we have only one restriction:

$$R\beta - r = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \beta_2$$

Further, recall that the estimated variance-covariance matrix of β is given by

$$\begin{aligned} \widehat{Var}(\beta) &= s^2(X'X)^{-1} = \\ & \begin{bmatrix} Var(\beta_1) & \ldots & \cdots \\ \cdots & Var(\beta_2) & \cdots \\ \cdots & \cdots & Var(\beta_3) \end{bmatrix} \end{aligned}$$

Since $R = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, it follows that

$$Rs^2(X'X)^{-1}R' = Var(\beta_2)$$

Thus

$$\begin{aligned} & (R\beta - r)'[Rs^2(X'X)^{-1}R']^{-1}(R\beta - r)/q \\ = \; & \frac{\beta_2^2}{Var(\beta_2)} \sim F(1, N-k) \end{aligned}$$

Note that $\frac{\beta_2^2}{Var(\beta_2)} = (\frac{\beta_2}{\widehat{se}(\beta_2)})^2$, and recall that $\frac{\beta_2}{\widehat{se}(\beta_2)} \sim t(N-k)$

EXAMPLE 2

We want to test the hypothesis: $b_2 + b_3 = 1$.

Recall that

$$\begin{aligned}\widehat{Var}(\beta) &= s^2(X'X)^{-1} \\ &= \begin{bmatrix} Var(\beta_1) & \ldots & \cdots \\ \cdots & Var(\beta_2) & \cdots \\ \cdots & \cdots & Var(\beta_3) \end{bmatrix}\end{aligned}$$

So we have only one restriction.

Notice that $R = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$. Thus

$$\begin{aligned}Rs^2(X'X)^{-1}R' &= Var(\beta_2) + Var(\beta_3) + 2Cov(\beta_2, \beta_3) \\ &= Var(\beta_2 + \beta_3)\end{aligned}$$

In addition $R\beta - r = \beta_2 + \beta_3 - 1$.

Finally,

$$\begin{aligned}& (R\beta - r)'[Rs^2(X'X)^{-1}R']^{-1}(R\beta - r)/q \\ = \; & \frac{(\beta_2 + \beta_3 - 1)^2}{Var(\beta_2 + \beta_3)} \sim F(1, N - k)\end{aligned}$$

Two other forms of the F test

$$\frac{\text{ESS}/(k-1)}{\text{RSS}/(N-k)} \sim F(k-1, N-k),$$

Since r^2 =ESS/TSS$= 1 - \frac{\text{RSS}}{\text{TSS}}$, the above equation yields

$$\frac{r^2/k-1}{(1-r^2)/N-k} \sim F(k-1, N-k)$$

Another form of the F test:

First, regress the restricted regression. For example, let the restriction be: $b_2 + b_3 = 1$ or $b_3 = 1 - b_2$

The unrestricted regression is

$$y_i = b_1 + b_2 x_{2i} + b_3 x_{3i} + e_i$$

whereas the restricted one is

$$\begin{aligned} y_i &= b_1 + b_2 x_{2i} - b_2 x_{3i} + x_{3i} + e_i \Rightarrow \\ (y_i - x_{3i}) &= b_1 + b_2 (x_{2i} - x_{3i}) + e_i \end{aligned}$$

The F test statistic is given by

$$\begin{aligned} \frac{(\mathsf{RSS}_R - \mathsf{RSS}_{UR})/q}{\mathsf{RSS}_{UR}/(N-k)} &\sim F(q, N-k), \text{ or} \\ \frac{(e'e_R - e'e_{UR})/q}{e'e_{UR}/(N-k)} &\sim F(q, N-k) \end{aligned}$$

PROOFS FROM WEEK 7 ONWARDS,

ONLY FOR EC5501

β, $R\beta$

Recall that

$$\beta \sim N[b, \sigma^2(X'X)^{-1}]$$

The expected value of $R\beta$ is: $E(R\beta) = Rb$

Next we want to calculate the variance-covariance matrix of $R\beta$:

$$\begin{aligned} Var(R\beta) &= E[(R\beta - Rb)(R\beta - Rb)'] \\ &= E[R(\beta - b)(\beta - b)'R'] \\ &= RE[(\beta - b)(\beta - b)']R' \\ &= RVar(\beta)R' = \underset{(qxk)}{R}\,\sigma^2\underset{(kxk)}{(X'X)^{-1}}\,\underset{(kxq)}{R'} \end{aligned}$$

Thus

$$R\beta \sim N[Rb, \sigma^2 R(X'X)^{-1}R']$$

and

$$(R\beta - Rb) \sim N[0, \sigma^2 R(X'X)^{-1}R']$$

Rewrite:

$$(R\beta - Rb) \sim N[0, \sigma^2 R(X'X)^{-1}R']$$

Under the null hypothesis, $H_0 : Rb = r$. Thus under the null

$$\underset{(qx1)}{(R\beta - r)} \sim N[0, \sigma^2 \underset{(qxq)}{R(X'X)^{-1}R'}]$$

$R\beta - r$ are the estimated restrictions

χ^2 DISTRIBUTION

We have q estimated restrictions $\underset{(qx1)}{R\beta - r}$. Each follows the N distribution.

Their sum of squares is given by $\underset{(1xq)}{(R\beta - r)'}\underset{(qx1)}{(R\beta - r)}$

Recall that $Var\underset{(qx1)}{(R\beta - r)} = \sigma^2 \underset{(qxq)}{R(X'X)^{-1}R'}$

The sum of squares, $\underset{(1xq)}{(R\beta - r)'}\underset{(qx1)}{(R\beta - r)}$, standardized by the variance of $R\beta - r$ is given by

$$\underset{(1xq)}{(R\beta - r)'}[\sigma^2 \underset{(qxq)}{R(X'X)^{-1}R'}]^{-1}\underset{(qx1)}{(R\beta - r)} \sim \chi^2(q)$$

and it follows the $\chi^2(q)$

F DISTRIBUTION

It can also be shown that the RSS divided by σ^2 follows the $\chi^2(N-k)$

$$\frac{\widehat{e}'\widehat{e}}{\sigma^2} \sim \chi^2(N-k)$$

It can also be shown that the ratio of two χ^2 distributions with q and $N-k$ degree of freedoms divided by $\frac{q}{N-k}$ follows the F distribution with q and $N-k$ degrees of freedom

That is

$$\frac{\underset{(1xq)}{(R\beta-r)'}\underset{(qxq)}{[\sigma^2 R(X'X)^{-1}R']^{-1}}\underset{(qx1)}{(R\beta-r)}}{\frac{\widehat{e}'\widehat{e}}{\sigma^2}\frac{q}{N-k}}$$

$$= \frac{\underset{(1xq)}{(R\beta-r)'}\underset{(qxq)}{[R(X'X)^{-1}R']^{-1}}\underset{(qx1)}{(R\beta-r)}}{s^2 q} \sim F(q, N-k)$$

since $\frac{\widehat{e}'\widehat{e}}{N-k} = s^2$.