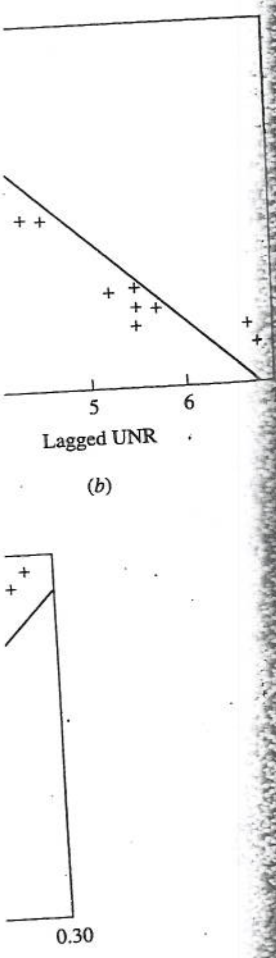

TABLE 2.2
Various inflation/unemployment regressions, 1957–1970*

Explanatory variable	Constant	Slope	r^2	S.E.R.
UNR	6.92 (3.82)	−0.8764 (−2.45)	0.33	1.40
UNR(−1)	9.13 (9.80)	−1.3386 (−7.19)	0.81	0.74
1/UNR(−1)	−4.48 (−6.51)	32.9772 (10.50)	0.90	0.54

*The t statistics are in parentheses. S.E.R. is the standard error of the regression.

through the rest of the 1960s. Figure 2.6*a* shows the scatter of inflation against the current unemployment rate. The slope is negative but the scatter is dispersed. In Fig. 2.6*b* inflation is plotted against the previous year's unemployment rate. A lagged response is not unreasonable since time is required for unemployment to affect wages and further time for wage changes to filter through to the prices of final goods. The scatter is now much tighter and there is an indication of nonlinearity. The same figure shows the fit of a *linear* regression of inflation on lagged unemployment. The linear specification is clearly an inadequate representation. Of the 14 residuals from the regression, 5 are positive and 9 are negative. The 5 positive residuals occur at the lowest and highest values of the explanatory variable. Inspection of the residuals can thus indicate possible misspecification. Runs of positive or negative residuals suggest misspecification.

Figure 2.6*c* shows the result of fitting the reciprocal relation

$$\text{INF} = \alpha + \gamma\left[\frac{1}{\text{UNR}(-1)}\right] + u \tag{2.17}$$

The residuals are somewhat smaller than in Fig. 2.6*b* and the scatter is more nearly linear, but not totally so. Table 2.2 summarizes the main results from the regressions associated with Fig. 2.6. We notice the substantial jump in r^2 on changing the explanatory variable from current to lagged unemployment, and a still further increase from 0.81 to 0.90 on using the reciprocal transformation.

Finally, we note the result of fitting the nonlinear relation

$$\text{INF} = \alpha_1 + \frac{\alpha_3}{\text{UNR}(-1) - \alpha_2} + u \tag{2.18}$$

This is fitted by nonlinear least squares, which is an iterative estimation process, commencing with some arbitrary values for the unknown parameters, calculating the residual sum of squares, then searching for changes in the parameters to reduce the RSS, and continuing in this way until successive changes in the estimated parameters and in the associated RSS are negligibly small. Standard errors and other test statistics can be produced at the final stage, just as in linear least squares; but, as will be explained shortly, they now have an asymptotic justification rather than exact, finite sample properties. As noted earlier the linearizing transformations obtained by setting α_1 or α_2 to zero impose theoretically inappropriate constraints on of wage (or price) inflation and unemployment can no longer be regarded as a serious piece of econometrics. However, in this chapter we are still restricted to two-variable relations, and the following example should only be taken as an illustration of the statistical steps in fitting nonlinear relations in two variables.

The data used are annual data for the United States from 1957 to 1970. The inflation variable (INF) is the annual percentage change in the Consumer Price Index (CPI). The unemployment variable (UNR) is the unemployment rate for civilian workers 16 years and over. Inflation ranges from a low of 0.69 percent in 1959 to a high of 5.72 percent in 1970, with a mean of 2.58 percent. The unemployment rate was 4.3 percent in 1957, rising to a peak of 6.7 percent in 1961 and falling steadily

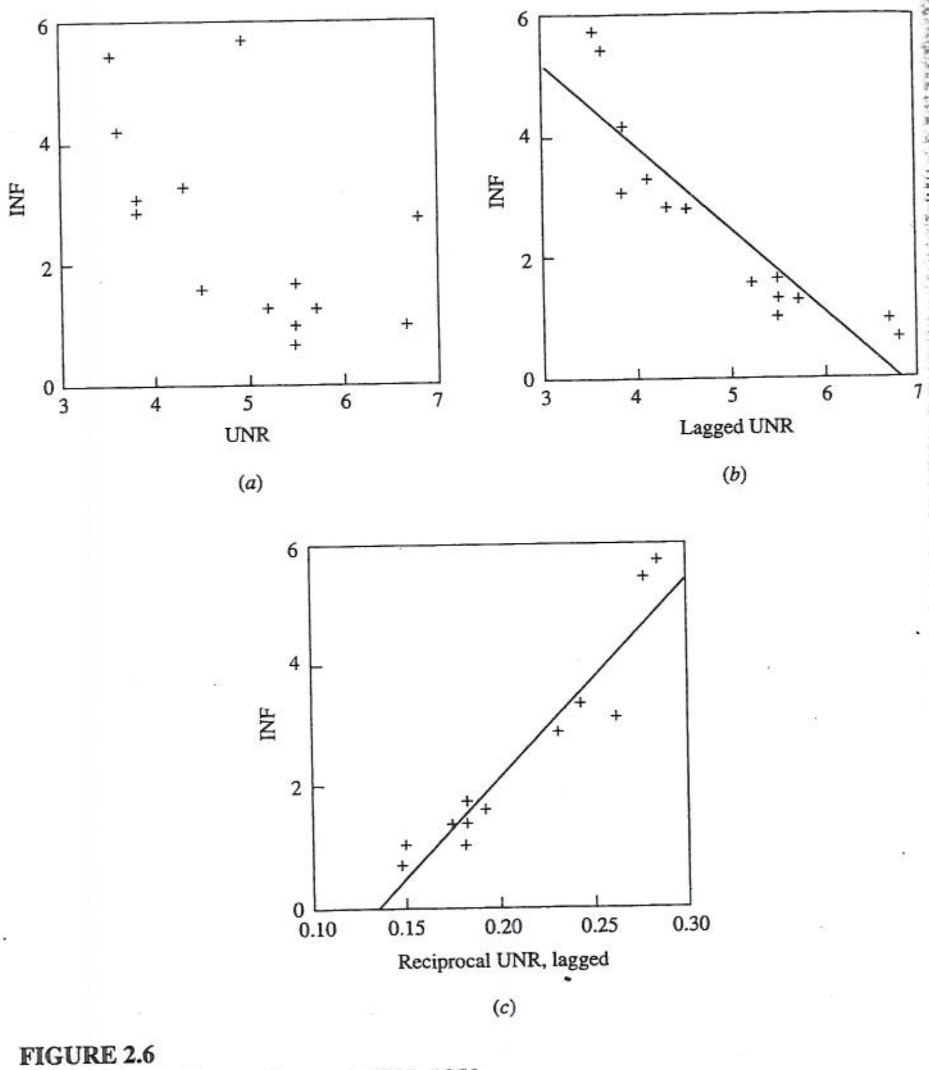

FIGURE 2.6
U.S. inflation and unemployment, 1957–1970.

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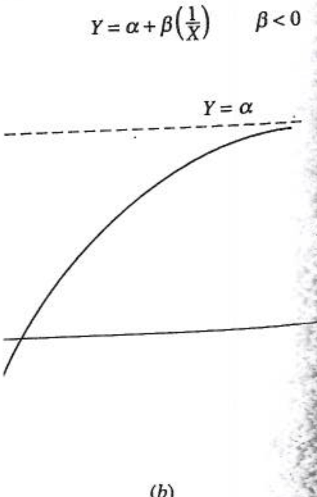


(b)

y nonlinear least squares.

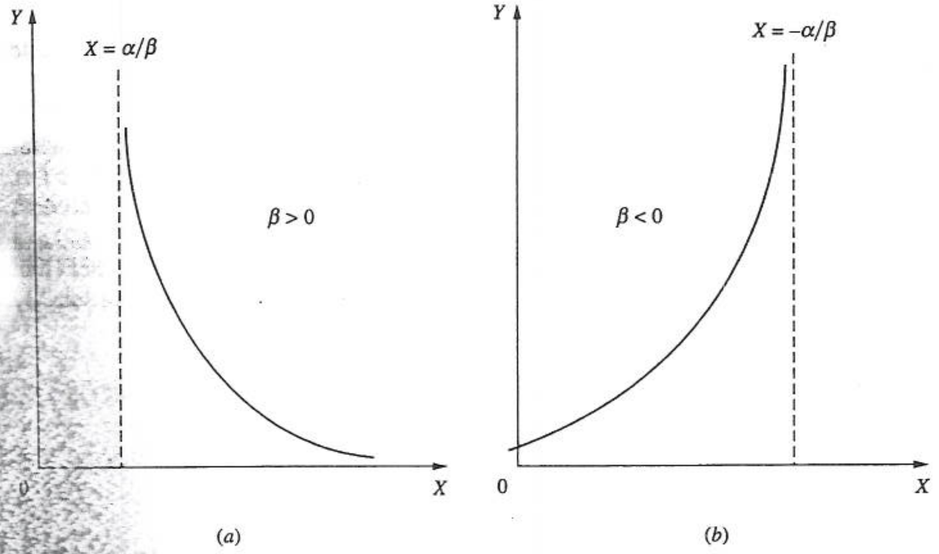


FIGURE 2.5
$\frac{1}{Y} = \alpha + \beta X$

The more general specification illustrated in Fig. 2.3*a* removes both restrictions and allows the possibility of a positive minimum rate of unemployment and a negative wage change. Figure 2.4*b* might represent a cross-section expenditure function. A certain threshold level of income is required before there is any expenditure on, say, restaurant meals, but such expenditure tends toward some upper limit, where the billionaire spends only infinitesimally more than the millionaire.

AN EMPIRICAL EXAMPLE OF A NONLINEAR RELATION: U.S. INFLATION AND UNEMPLOYMENT

The publication of the "Phillips curve" article in 1958 launched a new growth industry, whose practitioners searched for (and found) Phillips curves in various countries. In the original article Phillips plotted the annual percentage wage change in the United Kingdom against the unemployment rate for the period 1861 to 1913. The scatter revealed a negative nonlinear relation, which Phillips summarized in the form of a curved line. Most remarkably, data for two subsequent periods, 1913–1948, and 1948–1957, lay close to the curve derived from the 1861–1913 data. This simple Phillips curve has not survived the passage of time and has been subject to both statistical and theoretical attack and reformulation. Thus a simple two-variable analysis

A. W. Phillips, "The Relation between Unemployment and the Rate of Change of Money Wages in the United Kingdom, 1861–1957," *Economica*, New Series **25**, 1958, 283–299.

The result of adding an error term to Eq. (2.14) and attempting to minimize the residual sum of squares gives equations that are nonlinear in the α's. In this case there is no possible linearizing transformation that will take us back to the simple routines of Chapter 1.[5] However, there are two special cases of Eq. (2.14) where linearizing transformations are available. Setting α_2 to zero gives

$$Y = \alpha + \beta\left(\frac{1}{X}\right) \tag{2.15}$$

where $\alpha = \alpha_1$ and $\beta = \alpha_3$. Alternatively, setting α_1 to zero gives

$$\left(\frac{1}{Y}\right) = \alpha + \beta X \tag{2.16}$$

where $\alpha = -\alpha_2/\alpha_3$ and $\beta = 1/\alpha_3$. Illustrations are shown in Figs. 2.4 and 2.5.

Figure 2.4*a* has been fitted frequently in the study of **Phillips curves,** with Y representing the rate of wage or price change and X the unemployment rate. This specification carries the unrealistic implication that the asymptote for the unemployment rate is zero. The alternative simplification in Eq. (2.16) permits a positive minimum unemployment rate, but at the cost of imposing a zero minimum for wage change.

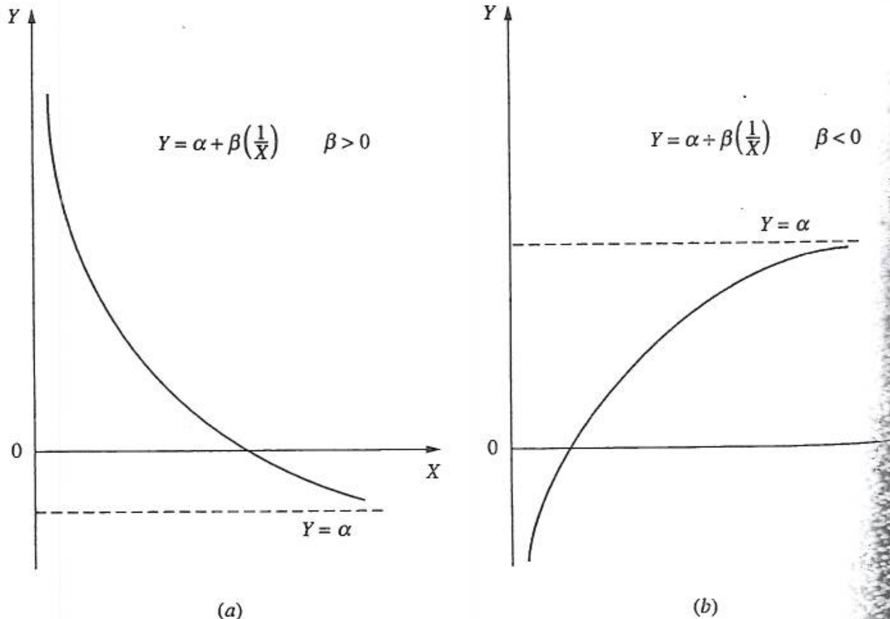


FIGURE 2.4
$Y = \alpha + \beta\left(\frac{1}{X}\right)$

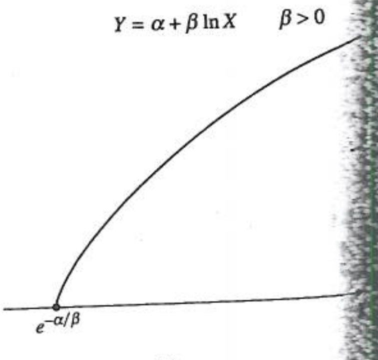
[5]As will be seen later, the equation may be fitted directly by nonlinear least squares.

nt growth equation. The general

(2.11)

nodels, where Y denotes earnings llows from Eq. (2.11) that

he *proportionate* change in Y *per* hown in Fig. 2.2*a*. Reversing the



(*b*)

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s in J. Mincer, *School, Experience, and*

axes gives

$$Y = \alpha + \beta \ln X \tag{2.12}$$

An illustration for positive β appears in Fig. 2.2*b*. In a cross-section study of household budgets such a curve might represent the relation between a class of expenditure Y and income X. A certain threshold level of income ($e^{-\alpha/\beta}$) is needed before anything is spent on this commodity. Expenditure then increases monotonically with income, but at a diminishing rate. The marginal propensity (β/X) to consume this good declines with increasing income, and the elasticity (β/Y) also declines as income increases.

2.2.3 Reciprocal Transformations

Reciprocal transformations are useful in modeling situations where there are **asymptotes** for one or both variables. Consider

$$(Y - \alpha_1)(X - \alpha_2) = \alpha_3 \tag{2.13}$$

This describes a **rectangular hyperbola** with asymptotes at $Y = \alpha_1$ and $X = \alpha_2$. Figure 2.3 shows some typical shapes for positive and negative α_3. Equation (2.13) may be rewritten as

$$Y = \alpha_1 + \frac{\alpha_3}{X - \alpha_2} \tag{2.14}$$

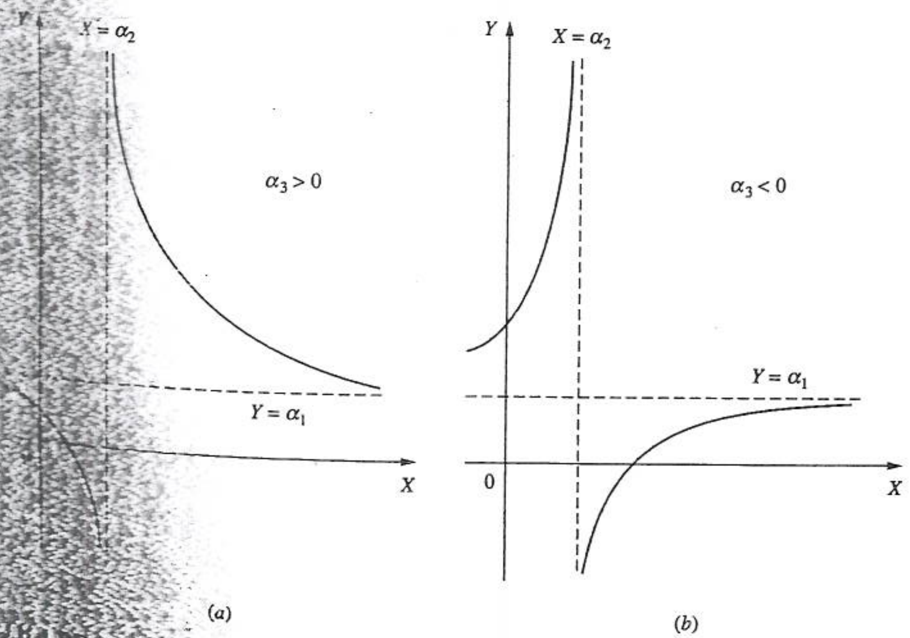

FIGURE 2.3
Rectangular hyperbola.

2.2.2 Semilog Transformations

One example has already been given in the constant growth equation. The general formulation is[3]

$$\ln Y = \alpha + \beta X + u \tag{2.11}$$

This specification is widely used in human capital models, where Y denotes earnings and X years of schooling or work experience.[4] It follows from Eq. (2.11) that

$$\frac{1}{Y}\frac{dY}{dX} = \beta$$

Thus the slope in the semilog regression estimates the *proportionate* change in Y *per unit* change in X. An illustration for positive β is shown in Fig. 2.2*a*. Reversing the

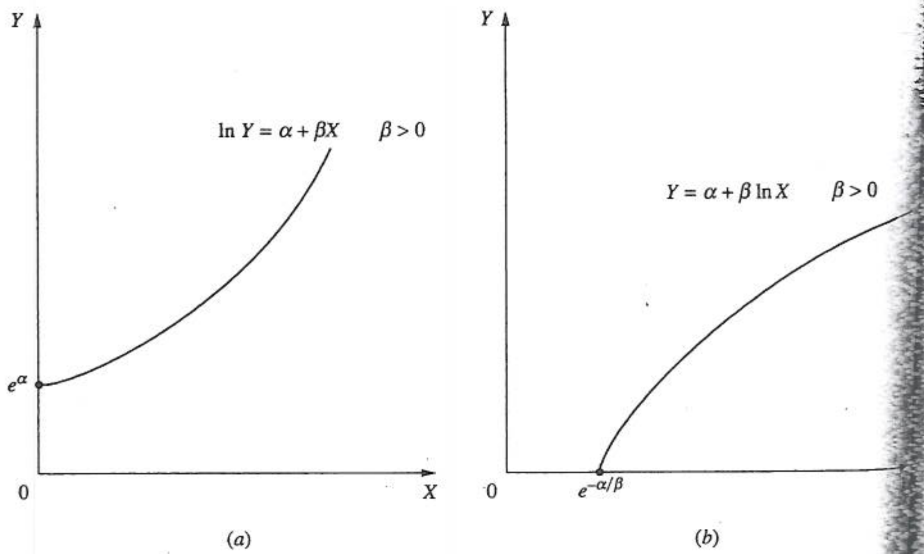

FIGURE 2.2
Semilog model.

[3]The astute reader will have noticed that in discussing various transformations we are playing fast loose with the disturbance term, inserting it in some equations and not in others in order to simplify transformations. The only justifications for such a (common) practice are ignorance and conven The late Sir Julian Huxley (distinguished biologist and brother of the novelist Aldous Huxley) described God as a "personified symbol for man's residual ignorance." The disturbance term pl similar role in econometrics, being a stochastic symbol for the econometrician's residual ignor And, just as one often does with God, one ascribes to the inscrutable and unknowable the prop most convenient for the purpose at hand.

[4]This specification is derived from theoretical considerations in J. Mincer, *School, Experience, and* ings, Columbia University Press, New York, 1974.

es, 1841–1910

Y	t	t(ln Y)
159	−3	−22.5457
904	−2	−16.9809
263	−1	−9.4263
926	0	0
238	1	11.3238
9081	2	23.8161
6853	3	38.0558
7424	0	24.2408

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dependent variable, the regressor variable, or both. Their main purpose is to achieve a **linearizing transformation** so that the simple techniques of Chapter 1 may be applied to suitably transformed variables and thus obviate the need to fit more complicated relations.

2.2.1 Log-Log Transformations

The growth equation has employed a transformation of the dependent variable. Many important econometric applications involve the logs of both variables. The relevant functional specification is

$$Y = AX^{\beta} \qquad \text{or} \qquad \ln Y = \alpha + \beta \ln X \tag{2.10}$$

where $\alpha = \ln A$. The **elasticity** of Y with respect to X is defined as

$$\text{Elasticity} = \frac{dY}{dX}\frac{X}{Y}$$

It measures the percent change in Y for a 1 percent change in X. Applying the elasticity formula to the first expression in Eq. (2.10) shows that the elasticity of this function is simply β, and the second expression in Eq. (2.10) shows that the slope of the log-log specification is the elasticity. Thus Eq. (2.10) specifies a **constant elasticity function.** Such specifications frequently appear in applied work, possibly because of their simplicity and ease of interpretation, since slopes in log-log regressions are direct estimates of (constant) elasticities. Figure 2.1 shows some typical shapes in the Y,X plane for various βs.

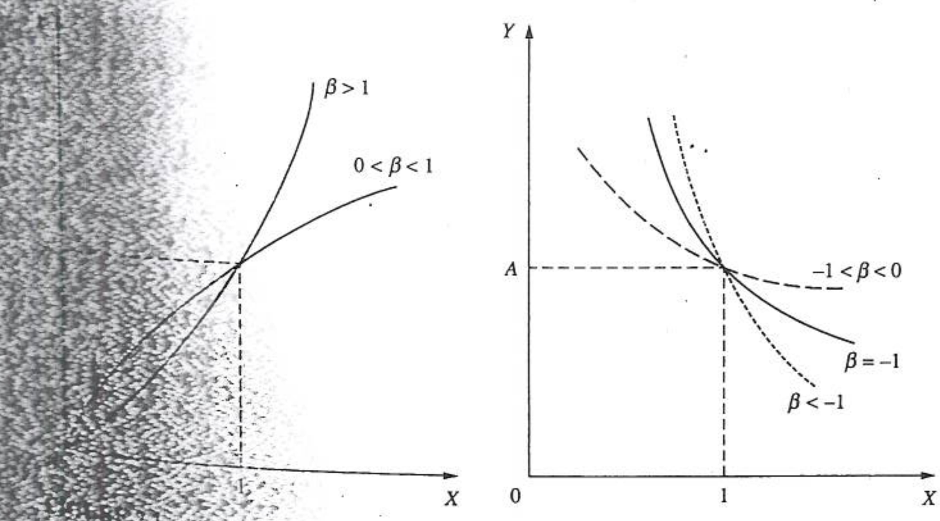

TABLE 2.1
Bituminous coal output in the United States, 1841–1910

Decade	Average annual output (1,000 net tons), Y	$\ln Y$	t	$t(\ln Y)$
1841–1850	1,837	7.5159	−3	−22.5457
1851–1860	4,868	8.4904	−2	−16.9809
1861–1870	12,411	9.4263	−1	−9.4263
1871–1880	32,617	10.3926	0	0
1881–1890	82,770	11.3238	1	11.3238
1891–1900	148,457	11.9081	2	23.8161
1901–1910	322,958	12.6853	3	38.0558
Sum		71.7424	0	24.2408

So we will fit a constant growth curve and estimate the annual growth rate. Setting the origin for time at the center of the 1870s and taking a unit of time to be 10 years, we obtain the t series shown in the table. From the data in the table

$$a = \frac{\sum \ln Y}{n} = \frac{71.7424}{7} = 10.2489$$

$$b = \frac{\sum t \ln Y}{\sum t^2} = \frac{24.2408}{28} = 0.8657$$

The r^2 for this regression is 0.9945, confirming the linearity of the scatter. The estimated growth rate per decade is obtained from

$$\hat{g} = e^b - 1 = 1.3768$$

Thus the constant growth rate is almost 140 percent per decade. The annual growth rate (agr) is then found from

$$(1 + \text{agr})^{10} = 2.3768$$

which gives agr $= 0.0904$, or just over 9 percent per annum. The equivalent continuous rate is 0.0866.

The time variable may be treated as a fixed regressor, and so the inference procedures of Chapter 1 are applicable to equations like (2.5) and (2.7).[2]

2.2 TRANSFORMATIONS OF VARIABLES

The log transformation of the dependent variable in growth studies leads naturally to the consideration of other transformations. These transformations may be of the

[2]For a very useful discussion of the use of time as a regressor, see Russell Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics,* Oxford University Press, 1993, pp. 115–118.