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A NEW METHOD FOR OBTAINING THE AUTOCOVARIANCE OF AN ARMA MODEL: AN EXACT FORM SOLUTION

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In this article we present a new method for computing the theoretical autocovariance function of an autoregressive moving average model. The importance of our theorem is that it yields two interesting results: First, a closed-form solution is derived in terms of the roots of the autoregressive polynomial and the parameters of the moving average part. Second, a sufficient condition for the lack of model redundancy is obtained.

1. INTRODUCTION

In this paper we present a new method for computing the theoretical autocovariance function of an autoregressive moving average (ARMA) model.

In Section 2 of this paper we give a new method for computing the theoretical autocovariance function of an autoregressive (AR) scheme. Exact methods of calculating the autocovariance for autoregressive models are given by Quenouille (1947) and Pagano (1973). We believe that our method is an improvement over those proposed by Quenouille and Pagano. It is exact, easily coded, and can be used for AR models of all orders.

In Section 3 of this paper we give a new method for computing the autocovariance function of the following ARMA scheme:

$$y_t = a_0 + \sum_{i=1}^{p} \phi_i^* y_{t-i} - \sum_{i=0}^{q} \theta_i \epsilon_{t-i}, \qquad \theta_0 = -1$$

(From now on, we will assume that $a_0 = 0$ for ease of calculation.) McLeod (1975) gives an algorithm for the computation of the autocovariances of the preceding process in terms of the parameters of the model $\phi_1^*, \ldots, \phi_p^*, \theta_1, \ldots, \theta_q$. In our method we express the autocovariance as an explicit function of the roots of

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the AR polynomial $(-\sum_{i=0}^{p} \phi_i^* L^i,\ \phi_0^* = -1)$ and the parameters of the moving average (MA) part $\theta_1, \theta_2, \ldots, \theta_q$. The only restriction that we impose is that all the roots (complex or real) of the AR polynomial are distinct. Nevertheless, the cases of two, three, and four equal roots are presented in the Appendixes.

To point out the importance of our theorem we quote Granger and Newbold (1986): "A specific form for the autocovariance sequence $\text{cov}_j(y_t)$ of an ARMA(p,q) model is more difficult to find than it was for the AR and MA models. In principle an expression for $\text{cov}_j(y_t)$ can be found by solving the difference equation

$$\sum_{i=0}^{p} \phi_i^* E(y_{t-j}\epsilon_{t-i}) = -\theta_j$$

for the $E(y_{t-j}\epsilon_{t-i})$, substituting into

$$g_j = -\sum_{i=0}^{q} \theta_i E(y_{t-j}\epsilon_{t-i})$$

to find the g's and then solving the difference equation

$$\sum_{i=0}^{p} \phi_i^* \,\text{cov}_{j-i}(y_t) = g_j$$

for the $\text{cov}_j(y_t)$. Because in general the results will be rather complicated, the eventual solution is not presented."

The reasons we present our proof are that it is simpler than the one based on solving differences equations and that it has two important consequences:

(a) It is convenient for obtaining a solution in closed form.
(b) It provides a sufficient condition for the lack of model redundancy for an ARMA process.[1]

2. AUTOCOVARIANCE OF AN AR(p)

Let y_t be an AR(2) process that is given by

$$y_t = \phi_1^* y_{t-1} + \phi_2^* y_{t-2} + \epsilon_t \Rightarrow (1 - \phi_1^* L - \phi_2^* L^2) y_t = \epsilon_t$$
$$\Rightarrow (1 - \phi_1 L)(1 - \phi_2 L) y_t = \epsilon_t, \qquad \epsilon_t \sim \text{i.i.d.}\,(0, \sigma^2). \tag{2.1}$$

(Without loss of generality, we assume that the variance of ϵ_t is 1.)

PROPOSITION 1. *The autocovariance function of the preceding* AR(2) *process is given by*

$$\text{cov}_n(y_t) = \frac{1}{(1 - \phi_1\phi_2)(\phi_1 - \phi_2)} \left(\frac{\phi_1^n \phi_1}{1 - \phi_1^2} - \frac{\phi_2^n \phi_2}{1 - \phi_2^2} \right). \tag{2.2}$$

Proof. The AR(2) process can be written as an AR(1) process with an AR(1) error term (z_t):

$$y_t = \phi_1 y_{t-1} + z_t \tag{2.3}$$

where $z_t = \phi_2 z_{t-1} + \epsilon_t$. The nth $(n \geq 1)$ autocovariance of y_t is given by

$$\begin{aligned} y_t &= \sum_{i=0}^{n-1} \phi_1^i z_{t-i} + \phi_1^n y_{t-n} \Rightarrow \operatorname{cov}(y_t, y_{t-n}) \\ &= \phi_1^n \operatorname{var}(y_t) + \sum_{i=0}^{n-1} \phi_1^i \operatorname{cov}(z_{t-i}, y_{t-n}). \end{aligned} \tag{2.4}$$

Because $y_{t-n} = \phi_1 y_{t-n-1} + z_{t-n} = \sum_{i=0}^{\infty} \phi_1^i z_{t-n-i}$ we have

$$\operatorname{cov}(z_t, y_{t-n}) = \sum_{i=0}^{\infty} \phi_1^i \operatorname{cov}(z_t, z_{t-n-i}). \tag{2.5}$$

Because $\operatorname{cov}(z_t, z_{t-j}) = \phi_2^j \operatorname{var}(z_t)$, we get

$$\operatorname{cov}(z_t, y_{t-n}) = \phi_2^n \operatorname{var}(z_t) \sum_{i=0}^{\infty} (\phi_1 \phi_2)^i = \frac{\phi_2^n \operatorname{var}(z_t)}{1 - \phi_1 \phi_2}. \tag{2.6}$$

From (2.4) and (2.6) we get

$$\begin{aligned} \operatorname{cov}_n(y_t) &= \phi_1^n \operatorname{var}(y_t) + \frac{\operatorname{var}(z_t) \sum_{i=0}^{n-1} \phi_2^{n-i} \phi_1^i}{(1 - \phi_1 \phi_2)} \\ &= \phi_1^n \operatorname{var}(y_t) + \frac{\phi_2^n}{1 - \phi_1 \phi_2} \operatorname{var}(z_t) \left[\frac{1 - (\phi_1/\phi_2)^n}{1 - (\phi_1/\phi_2)} \right]. \end{aligned} \tag{2.7}$$

Using (2.3) and (2.6) we can write the variance of y_t as

$$\operatorname{var}(y_t) = \frac{\operatorname{var}(z_t) + 2\phi_1 \phi_2 \dfrac{\operatorname{var}(z_t)}{1 - \phi_1 \phi_2}}{1 - \phi_1^2} = \frac{1 + \phi_1 \phi_2}{(1 - \phi_1 \phi_2)(1 - \phi_1^2)(1 - \phi_2^2)}. \tag{2.8}$$

From equations (2.7) and (2.8) we have

$$\operatorname{cov}_n(y_t) = \frac{1}{(1 - \phi_1 \phi_2)(\phi_1 - \phi_2)} \left(\frac{\phi_1^n \phi_1}{1 - \phi_1^2} - \frac{\phi_2^n \phi_2}{1 - \phi_2^2} \right). \tag{2.9}$$

■

Important Property. Equation (2.3) is symmetric with respect to ϕ_1 and ϕ_2. Hence the autocovariance function (2.9) is symmetric with respect to ϕ_1 and ϕ_2.

We note that the nth autocovariance of an AR(2) process is a function of the inverse of the two roots (hereafter i-roots) ϕ_1 and ϕ_2 of the second-order polynomial $(1-\phi_1^* L - \phi_2^* L^2)$ and can be expressed as the sum of two terms. The first term is the product of the first i-root raised to the power of n (ϕ_1^n) and a coefficient that depends on the two i-roots $\{\phi_1/[(1-\phi_1\phi_2)(\phi_1-\phi_2)(1-\phi_1^2)]\}$. The second term is the product of the second i-root raised to the power of n (ϕ_2^n) and a coefficient that depends on the two i-roots $\{\phi_2/[(1-\phi_1\phi_2)(\phi_2-\phi_1)(1-\phi_2^2)]\}$. Given the symmetry of the autocovariance function in the i-roots of the AR polynomial the coefficients of the two terms are such that if in the first term we interchange ϕ_1 and ϕ_2 we will get the second one. The preceding form of the autocovariance of an AR(2) process leads us to the first principal result of this paper.

Let y_t be an AR(p) process that is given by

$$y_t = \sum_{i=1}^{p} \phi_i^* y_{t-i} + \epsilon_t \Rightarrow \prod_{i=1}^{p} (1-\phi_i L) y_t = \epsilon_t. \tag{2.10}$$

THEOREM 1. *The autocovariance function of the preceding* AR(p) *process is given by*

$$\mathrm{cov}_n(y_t) = \sum_{j=1}^{p} \frac{\phi_j^{p-1}\phi_j^n}{\prod_{i=1}^{p}(1-\phi_i\phi_j) \prod_{(l=1,\, l\neq j)}^{p} (\phi_j - \phi_l)} = \sum_{j=1}^{p} e_{jn}. \tag{2.11}$$

Proof (1st Method). We prove the theorem by induction: In Proposition 1 we proved that it holds for an AR(2) process; if we assume that it holds for an AR$(p-1)$ process, then it will be sufficient to prove that it holds for an AR(p).

Let y_t be an AR(p) process. Then it can be written as an AR(1) process with an AR$(p-1)$ error term $[y_t = \phi_1 y_{t-1} + z_t$ where z_t is an AR$(p-1)$ process given by$]$

$$\prod_{i=2}^{p} (1-\phi_i L) z_t = \epsilon_t. \tag{2.12}$$

Notice that the AR(p) process is symmetric with respect to the p i-roots of the AR polynomial $(\phi_1, \ldots, \phi_p)$. Hence, the autocovariance function will also be symmetric with respect to the p i-roots of the AR polynomial.

For easy reference we rewrite (2.4) and (2.5):

$$\mathrm{cov}_n(y_t) = \phi_1^n \,\mathrm{var}(y_t) + \sum_{i=0}^{n-1} \phi_1^i \,\mathrm{cov}(z_{t-i}, y_{t-n}), \tag{2.13}$$

$$\mathrm{cov}(z_t, y_{t-n}) = \sum_{i=0}^{\infty} \phi_1^i \,\mathrm{cov}(z_t, z_{t-n-i}). \tag{2.14}$$

In equation(2.13), when $n = 0$, the lower limit exceeds the upper limit of the summation. When this is the case we will say that the summation vanishes. In other words for $n = 0$ equation (2.13) becomes a tautology.

Because z_t is an AR$(p-1)$ process, its autocovariance is given by

$$\text{cov}_n(z_t) = \sum_{j=2}^{p} \frac{\phi_j^{p-2}\phi_j^n}{\prod_{i=2}^{p}(1-\phi_j\phi_i)\prod_{(l=2,l\neq j)}^{p}(\phi_j-\phi_l)} = \sum_{j=2}^{p}\hat{e}_{jn}. \tag{2.15}$$

Substituting (2.15) into (2.14) we get

$$\begin{aligned}\text{cov}(z_t, y_{t-n}) &= \sum_{j=2}^{p} \frac{\phi_j^{p-2}\phi_j^n}{\prod_{i=2}^{p}(1-\phi_j\phi_i)\prod_{(l=2,l\neq j)}^{p}(\phi_j-\phi_l)(1-\phi_1\phi_j)} \\ &= \sum_{j=2}^{p} \frac{\phi_j^{p-2}\phi_j^n}{\prod_{i=1}^{p}(1-\phi_j\phi_i)\prod_{(l=2,l\neq j)}^{p}(\phi_j-\phi_l)}. \end{aligned} \tag{2.16}$$

Substituting (2.16) into (2.13) we get

$$\begin{aligned}\text{cov}_n(y_t) &= \phi_1^n \,\text{var}(y_t) + \sum_{j=2}^{p} \frac{\phi_j^{p-1}\phi_j^n \dfrac{1-(\phi_1/\phi_j)^n}{\phi_j-\phi_1}}{\prod_{i=1}^{p}(1-\phi_j\phi_i)\prod_{(l=2,l\neq j)}^{p}(\phi_j-\phi_l)} \\ &= \phi_1^n A + \sum_{j=2}^{p}\phi_j^n e_{j0}, \end{aligned} \tag{2.17}$$

where

$$A = \text{var}(y_t) - \sum_{j=2}^{p} \frac{\phi_j^{p-1}}{\prod_{i=1}^{p}(1-\phi_j\phi_i)\prod_{(l=1,l\neq j)}^{p}(\phi_j-\phi_l)}. \tag{2.17a}$$

The autocovariance function is the sum of p terms. The first term is the product of ϕ_1^n and a coefficient that depends on $\phi_1,\phi_2,\ldots,\phi_p,(A)$. Each of the rest of the $p-1$ terms is the product of ϕ_j^n where $j=2,3,\ldots,p$ and a coefficient that depends on $\phi_1,\ldots,\phi_p,(e_{j0})$. Given the symmetry of the autocovariance function in the i-roots of the pth-order AR polynomial, if we interchange the inverse of the j-root ϕ_j with the inverse of the first root ϕ_1 in the coefficient e_{j0} we will get the coefficient of ϕ_1^n, that is, coefficient A. Hence, A is given by

$$A = \frac{\phi_1^{p-1}}{\prod_{i=1}^{p}(1-\phi_1\phi_i)\prod_{l=2}^{p}(\phi_1-\phi_l)} = e_{10}. \tag{2.18}$$

Finally, substituting the preceding equation into (2.17) we get equation (2.11). ■

Proof (2nd Method). The preceding proof does not hold for the variance of y_t, because, as we mentioned earlier on, when $n = 0$, equation (2.13) becomes a tautology. However, there is an alternative proof that holds for the variance as well.

The covariance between y_t and z_{t-n} is given by

$$y_t = \phi_1 y_{t-1} + z_t = \sum_{i=0}^{n-1} \phi_1^i z_{t-i} + \sum_{i=0}^{\infty} \phi_1^{n+i} z_{t-n-i} \Rightarrow \operatorname{cov}(y_t, z_{t-n})$$

$$= \sum_{i=0}^{n-1} \phi_1^i \operatorname{cov}_{n-i}(z_t) + \sum_{i=0}^{\infty} \phi_1^{n+i} \operatorname{cov}_i(z_t). \tag{2.19}$$

Substituting (2.15) into (2.19) we get

$$\operatorname{cov}(y_t, z_{t-n}) = \sum_{j=2}^{p} \hat{e}_{j,n+1} \frac{1 - \left(\dfrac{\phi_1}{\phi_j}\right)^n}{\phi_j - \phi_1} + \phi_1^n \sum_{j=2}^{p} \frac{\hat{e}_{j0}}{1 - \phi_1 \phi_j}$$

$$= \sum_{j=2}^{p} \frac{\hat{e}_{j1} \phi_j^n}{\phi_j - \phi_1} + \phi_1^n \left(\sum_{j=2}^{p} \frac{\hat{e}_{j0}}{1 - \phi_1 \phi_j} - \sum_{j=2}^{p} \frac{\hat{e}_{j1}}{\phi_j - \phi_1} \right). \tag{2.20}$$

Note that, when $n = 0$, the first term in the right-hand side of the preceding equation becomes zero, and this is in accordance with the fact that, when $n = 0$, the lower limit exceeds the upper limit of the summation of the first term in the right-hand side of equation (2.19).

The nth ($n \geq 0$) autocovariance of y_t is given by

$$y_{t-n} = \sum_{i=0}^{\infty} \phi_1^i z_{t-n-i} \Rightarrow \operatorname{cov}_n(y_t) = \sum_{i=0}^{\infty} \phi_1^i \operatorname{cov}(y_t, z_{t-n-i}). \tag{2.21}$$

Substituting (2.20) into (2.21), and after some algebra, we get

$$\operatorname{cov}_n(y_t) = \sum_{j=2}^{p} \phi_j^n e_{j0} + \phi_1^n \hat{A}, \tag{2.22}$$

where

$$\hat{A} = \frac{1}{1 - \phi_1^2} \left[\sum_{j=2}^{p} \frac{\hat{e}_{j0}}{(1 - \phi_1 \phi_j)} - \sum_{j=2}^{p} \frac{\hat{e}_{j1}}{\phi_j - \phi_1} \right]. \tag{2.22a}$$

We employ the same reasoning with the one used in the first method (p. 626) to get

$$\hat{A} = e_{10}. \tag{2.23}$$

Finally, substituting the preceding equation into (2.22) we get equation (2.11). ■

3. AUTOCOVARIANCE OF AN ARMA(p,q)

Let z_t be an ARMA$(1,q)$ process given by

$$z_t = \phi_2 z_{t-1} - \sum_{i=0}^{q} \theta_i \epsilon_{t-i}, \tag{3.1}$$

where $\theta_0 = -1$ and $\epsilon_t \sim$ i.i.d. $(0,\sigma^2)$.[2]

PROPOSITION 2. *The autocovariance function of z_t is given by*

$$\text{cov}_j(z_t) = \begin{cases} \dfrac{\phi_2^j}{1-\phi_2^2}\lambda_{2j}, & j \le q-1 \\ \dfrac{\phi_2^j}{1-\phi_2^2}\lambda_{2q}, & j \ge q, \end{cases} \tag{3.2}$$

where

$$\lambda_{2j} = \sum_{i=0}^{q} \theta_i^2 + \sum_{l=1}^{j}\sum_{k=0}^{q-l} \theta_k\theta_{k+l}(\phi_2^l + \phi_2^{-l}) + \sum_{l=j+1}^{q}\sum_{k=0}^{q-l} \theta_k\theta_{k+l}(\phi_2^l + \phi_2^{l-2j}). \tag{3.2a}$$

Proof. It is not difficult to show that

$$\text{cov}_j(z_t) = \phi_2^j \,\text{var}(z_t) + \sum_{i=1}^{\min(j,q)} a_i \phi_2^{j-i}, \tag{3.3}$$

where

$$a_i = -\sum_{m=0}^{q-i} \theta_{i+m}\phi_2^m + \sum_{n=0}^{q-1-i}\sum_{k=1}^{q-i-n} \theta_k\theta_{n+i+k}\phi_2^n. \tag{3.3a}$$

It should be noted that, for $i = q$, the preceding double summation vanishes because the lower limit exceeds the upper limit of the summation operator.

It can be seen that the variance of z_t is given by

$$(1-\phi_2^2)\text{var}(z_t) = \sum_{m=0}^{q} \theta_m^2 + 2\phi_2\left(-\sum_{n=1}^{q} \phi_2^{n-1}\theta_n + \sum_{l=1}^{q-1}\sum_{k=1}^{q-l} \theta_k\theta_{k+l}\phi_2^{l-1}\right). \tag{3.4}$$

We substitute equations (3.3a) and (3.4) into (3.3) and (after some algebra) we get equation (3.2). ■

Theorem 1 and Proposition 2 lead us to the second principal result of this paper.

Let y_t be an ARMA(p,q) process given by

$$y_t = \sum_{i=1}^{p} \phi_i^* y_{t-i} - \sum_{i=0}^{q} \theta_i \epsilon_{t-i}, \tag{3.5}$$

where $\theta_0 = -1$ and $\epsilon_t \sim$ i.i.d.$(0,1)$.

THEOREM 2. *The autocovariance function of the preceding* ARMA(p,q) *process* (y_t) *is given by*

$$\text{cov}_j(y_t) = \begin{cases} \sum_{i=1}^{p} e_{ij}\,\lambda_{ij}, & j \le q-1 \\ \sum_{i=1}^{p} e_{ij}\,\lambda_{iq}, & j \ge q, \end{cases} \tag{3.6}$$

where

$$e_{ij} = \frac{\phi_i^j \phi_i^{p-1}}{\prod_{l=1}^{p}(1-\phi_l\phi_i)\prod_{(k=1,\,k\neq i)}^{p}(\phi_i-\phi_k)} \tag{3.6a}$$

and

$$\lambda_{ij} = \sum_{k=0}^{q}\theta_k^2 + \sum_{l=1}^{j}\sum_{k=0}^{q-l}\theta_k\theta_{k+l}(\phi_i^l+\phi_i^{-l}) + \sum_{l=j+1}^{q}\sum_{k=0}^{q-l}\theta_k\theta_{k+l}(\phi_i^l+\phi_i^{l-2j}). \tag{3.6b}$$

Note that, for $j = q$, *the third term of the right-hand side of the preceding equation disappears because the lower limit exceeds the upper limit of the first summation operator.*

Proof (Case i, $j \ge q$). We prove the theorem by induction: In Proposition 2 we proved that the theorem holds for an ARMA$(1,q)$ process; if we assume that it holds for an ARMA$(p-1,q)$ process, then it will be sufficient to prove that it holds for an ARMA(p,q) process.

Let y_t be an ARMA(p,q) process. Then it can be written as an AR(1) process with an ARMA$(p-1,q)$ error term $(y_t = \phi_1 y_{t-1} + z_t)$, where z_t is an ARMA $(p-1,q)$ process given by

$$\prod_{i=2}^{p}(1-\phi_i L)z_t = -\sum_{i=0}^{q}\theta_i\epsilon_{t-i}. \tag{3.7}$$

Notice that the ARMA(p,q) process is symmetric with respect to the i-roots of the pth-order AR polynomial. Hence, the autocovariance will also be symmetric with respect to the i-roots of the AR polynomial. Because z_t is an ARMA$(p-1,q)$ process, its autocovariance is given by

$$\text{cov}_j(z_t) = \begin{cases} \sum_{i=2}^{p}\hat{e}_{ij}\,\lambda_{ij} = \sum_{i=2}^{p}\hat{e}_{ij}(\lambda_{iq}+\nu_{ij}), & 0 \le j \le q-1 \\ \sum_{i=2}^{p}\hat{e}_{ij}\,\lambda_{iq}, & j \ge q, \end{cases} \tag{3.8}$$

where

$$\hat{e}_{ij} = \frac{\phi_i^j \phi_i^{p-2}}{\prod_{l=2}^{p}(1-\phi_l\phi_i) \prod_{(k=2,k\neq i)}^{p}(\phi_i-\phi_k)} \tag{3.8a}$$

$$\nu_{ij} = -\sum_{m=0}^{q-j-1}\theta_{q-m}(\phi_i^{q-2j-m}-\phi_i^{-q+m}) + \sum_{k=1}^{q-1-j}\sum_{l=0}^{q-j-(k+1)}\theta_k\theta_{q-l}(\phi_i^{q-2j-(l+k)}-\phi_i^{-q+l+k}) \tag{3.8b}$$

and λ_{iq} is given by (3.6b). Note that, when $j = q-1$, the second term in the preceding equation vanishes because the lower limit exceeds the upper limit of the summation operator. Moreover, the relation between the $\hat{e}$'s is $\hat{e}_{ij} = \phi_i \hat{e}_{i,j-1} = \cdots = \phi_i^{j-1}\hat{e}_{i1}$.

In equation (3.8) we expressed the λ_{ij} coefficients as functions of the λ_{iq} coefficients. The covariance of z_t and y_{t-j} (for $1 \leq j \leq q-1$) is given by

$$\text{cov}(z_t, y_{t-j}) = \sum_{i=0}^{\infty}\phi_1^i \,\text{cov}(z_t, z_{t-(j+i)}). \tag{3.9}$$

Substituting (3.8) into (3.9) we get

$$\text{cov}(z_t, y_{t-j}) = \sum_{i=2}^{p}\frac{\hat{e}_{ij}}{1-\phi_1\phi_i}\lambda_{iq} + \pi\sum_{i=2}^{p}\hat{e}_{ij} f_{ij}, \tag{3.10}$$

where

$$f_{ij} = \sum_{v=j}^{q-1}\nu_{iv}(\phi_1\phi_i)^{v-j}, \tag{3.10a}$$

and $\pi = 0$, for $j \geq q$, $\pi = 1$, for $1 \leq j \leq q-1$.

The jth autocovariance of y_t is given by

$$\begin{aligned}\text{cov}_j(y_t) &= \sum_{l=0}^{j-1}\text{cov}(z_{t-l}, y_{t-j})\phi_1^l + \phi_1^j \,\text{var}(y_t) \\ &= \sum_{v=1}^{j}\text{cov}(z_t, y_{t-v})\phi_1^{j-v} + \phi_1^j \,\text{var}(y_t).\end{aligned} \tag{3.11}$$

Substituting (3.10) into the preceding equation, and after some algebra, we get

$$\begin{aligned}\text{cov}_j(y_t) &= \sum_{i=2}^{p} e_{i0}\lambda_{iq}\phi_i^j\left[1-\left(\frac{\phi_1}{\phi_i}\right)^j\right] + \phi_1^j \,\text{var}(y_t) + \sum_{v=1}^{q-1}\sum_{i=2}^{p}\phi_1^{j-v}\hat{e}_{iv} f_{iv} \\ &= \sum_{i=2}^{p} e_{ij}\lambda_{iq} + \phi_1^j \zeta,\end{aligned} \tag{3.12}$$

where

$$\zeta = \mathrm{var}(y_t) - \sum_{i=2}^{p} e_{i0}\lambda_{iq} + \sum_{v=1}^{q-1}\sum_{i=2}^{p} \phi_1^{-v}\hat{e}_{iv} f_{iv}. \quad \textbf{(3.12a)}$$

The autocovariance of the ARMA(p,q) process is the sum of p terms. The last term is the product of ϕ_1^j and a coefficient that depends on $\phi_1,\ldots,\phi_p$ and $\theta_1,\ldots,\theta_q$. Each of the first $p-1$ terms is the product of ϕ_i^j where $i=2,\ldots,p$ and a coefficient that depends on $\phi_1,\ldots,\phi_p,\theta_1,\ldots,\theta_q(e_{i0}\lambda_{iq})$. Given the symmetry of the autocovariance function in the i-roots of the AR polynomial, if we interchange the i-root ϕ_i with the first i-root ϕ_1 in the coefficient $e_{i0}\lambda_{iq}$ we will get the coefficient of ϕ_1^j, that is, coefficient ζ. Hence, ζ is given by

$$\zeta = e_{10}\lambda_{1q}. \quad \textbf{(3.13)}$$

Finally, substituting (3.13) into (3.12) we get (3.6). ■

Proof (Case ii, $1 \le j \le q-1$, 1st Method). Because z_t is an ARMA$(p-1,q)$ process its autocovariance function is given by

$$\mathrm{cov}_n(z_t) = \begin{cases} \sum_{i=2}^{p}\hat{e}_{in}\lambda_{iq} = \sum_{i=2}^{p}\hat{e}_{in}(\lambda_{ij}-\nu_{ij}), & n \ge q \\ \sum_{i=2}^{p}\hat{e}_{in}\lambda_{i,j+b} = \sum_{i=2}^{p}\hat{e}_{in}(\lambda_{ij}+\lambda_{ij,b^+}), & n=j+b, \quad q-1-j \ge b \ge 1 \\ \sum_{i=2}^{p}\hat{e}_{in}\lambda_{ij}, & n=j \\ \sum_{i=2}^{p}\hat{e}_{in}\lambda_{i,j-b} = \sum_{i=2}^{p}\hat{e}_{in}(\lambda_{ij}+\lambda_{ij,b^-}), & n=j-b, \quad j \ge b \ge 1, \end{cases} \quad \textbf{(3.14)}$$

where $\hat{e}_{in}$ is given by (3.8a), ν_{ij} is given by (3.8b), and λ_{ij,b^+} and λ_{ij,b^-} are given by

$$\lambda_{ij,b^+} = \sum_{v=0}^{q-j-1}\left(-\theta_{q-v} + \sum_{l=1}^{v}\theta_l\theta_{q-(v-l)}\right)\left(\phi_i^{\pi_1(q-v)-2\pi_2(j+b)} - \phi_i^{q-2j-v}\right), \quad \textbf{(3.14a)}$$

where $\pi_1 = 1$ for $v \le q-j-b-1$, $\pi_1 = -1$ for $v \ge q-j-b$ and $\pi_2 = 1$ for $v \le q-j-b-1$, $\pi_2 = 0$ for $v \ge q-j-b$ and $1 \le j \le q-2$, $1 \le b \le q-1-j$, and

$$\lambda_{ij,b^-} = \sum_{v=0}^{q+b-(j+1)}\left(-\theta_{q-v} + \sum_{l=1}^{v}\theta_l\theta_{q-(v-l)}\right)\left(\phi_i^{q+2b-(2j+v)} - \phi_i^{\pi_1(q-v)-2\pi_2 j}\right), \quad \textbf{(3.14b)}$$

where $\pi_1 = 1$ for $0 \le v \le q-j-1$, $\pi_1 = -1$ for $v \ge q-j$ and $\pi_2 = 1$ for $0 \le v \le q-j-1$, $\pi_2 = 0$ for $v \ge q-j$ and $2 \le j \le q-1$, $1 \le b \le j-1$.

In equation (3.14) we express the λ_{iq}, $\lambda_{i,j-b}$, and $\lambda_{i,j+b}$ as functions of the λ_{ij} coefficients. Note that in equations (3.14a) and (3.14b), when $v=0$, the second summation term vanishes because the lower limit exceeds the upper limit of the summation operator.

The covariance between z_t and y_{t-v} is given by

$$\operatorname{cov}(z_t, y_{t-v}) = \sum_{i=0}^{\infty} \phi_1^i \operatorname{cov}_{v+i}(z_t). \tag{3.15}$$

Substituting (3.14) into (3.15), and after some algebra, we get

$$\operatorname{cov}(z_t, y_{t-v}) = \sum_{i=2}^{p} \frac{\hat{e}_{iv} \lambda_{ij}}{1 - \phi_1 \phi_i} + \sum_{i=2}^{p} \hat{e}_{iv} f_{iv}^*, \tag{3.16}$$

where

$$f_{iv}^* = \sum_{m=v}^{j-1} \lambda_{ij,(j-m)^-} (\phi_1 \phi_i)^{m-v} + \sum_{l=1}^{q-j-1} \lambda_{ij,l^+} (\phi_1 \cdot \phi_i)^{j+l-v} - \nu_{ij} \frac{(\phi_1 \phi_i)^{q-v}}{1 - \phi_1 \phi_i}. \tag{3.16a}$$

Note that in f_{iv}^*, when $v = j$, the first term becomes zero, and when $j = q - 1$, the second term becomes zero because the lower limit exceeds the upper limit of the corresponding summation operator.

The jth autocovariance of y_t is given by

$$\operatorname{cov}_j(y_t) = \sum_{v=1}^{j} \operatorname{cov}(z_t, y_{t-v}) \phi_1^{j-v} + \phi_1^j \operatorname{var}(y_t). \tag{3.17}$$

Substituting (3.16) into the preceding equation, and after some algebra, we get

$$\begin{aligned} \operatorname{cov}_j(y_t) &= \sum_{i=2}^{p} \frac{\hat{e}_{i1} \lambda_{ij}}{(1 - \phi_1 \phi_i)(\phi_i - \phi_1)} \phi_i^j \left[1 - \left(\frac{\phi_1}{\phi_i} \right)^j \right] + \phi_1^j \operatorname{var}(y_t) \\ &\quad + \sum_{v=1}^{j} \sum_{i=2}^{p} \phi_1^{j-v} \hat{e}_{iv} f_{iv}^* \\ &= \sum_{i=2}^{p} e_{ij} \lambda_{ij} + \phi_1^j \zeta^*, \end{aligned} \tag{3.18}$$

where

$$\zeta^* = \operatorname{var}(y_t) + \sum_{v=1}^{j} \sum_{i=2}^{p} \phi_1^{-v} \hat{e}_{iv} f_{iv}^* - \sum_{i=2}^{p} e_{i0} \lambda_{ij}. \tag{3.18a}$$

We employ the same reasoning with the one used for the $j \geq q$ case (p. 631) to get

$$\zeta^* = e_{10} \lambda_{1j}. \tag{3.19}$$

Finally, substituting the preceding equation into (3.18) we get (3.6). ■

Proof (Case ii, $0 \le j \le q-1$, 2nd Method). The covariance between y_t and $z_{t-(j+v)}$ is given by

$$y_t = \phi_1 y_{t-1} + z_t = \sum_{i=0}^{j+v-1} \phi_1^i z_{t-i} + \sum_{i=0}^{\infty} \phi_1^{j+v+i} z_{t-(j+v+i)} \Rightarrow \operatorname{cov}(y_t, z_{t-(j+v)})$$

$$= \sum_{i=0}^{j+v-1} \phi_1^i \operatorname{cov}_{j+v-i}(z_t) + \sum_{i=0}^{\infty} \phi_1^{j+v+i} \operatorname{cov}_i(z_t). \quad \textbf{(3.20)}$$

Substituting (3.14) into the preceding equation, and after some algebra, we get

$$\begin{aligned}
&\operatorname{cov}(y_t, z_{t-(j+v)}) \\
&= \sum_{i=2}^{p} \frac{\hat{e}_{i1}\lambda_{ij}}{(\phi_i - \phi_1)} (\phi_i^{j+v} - \phi_1^{j+v}) + \sum_{k=1}^{j-1}\sum_{i=2}^{p} \hat{e}_{i,(j-k)} \lambda_{ij,k^-} \phi_1^{k+v} \\
&\quad + \sum_{k=1}^{\min(v,q-1-j)} \sum_{i=2}^{p} \hat{e}_{i,(j+k)} \lambda_{ij,k^+} \phi_1^{v-k} \\
&\quad - \pi_1 \sum_{i=2}^{p} \frac{\nu_{ij}}{(\phi_i - \phi_1)} \hat{e}_{iq} (\phi_i^{v+j-q+1} - \phi_1^{v+j-q+1}) \\
&\quad + \phi_1^{j+v} \sum_{i=2}^{p} \frac{\hat{e}_{i0}\lambda_{ij}}{(1-\phi_1\phi_i)} + \phi_1^{j+v} \sum_{k=0}^{j-1}\sum_{i=2}^{p} \hat{e}_{i0} \lambda_{ij,(j-k)^-} (\phi_1\phi_i)^k \\
&\quad + \phi_1^{2j+v} \sum_{i=2}^{p}\sum_{k=1}^{q-1-j} \hat{e}_{ij} \lambda_{ij,k^+} (\phi_1\phi_i)^k - \phi_1^{q+j+v} \sum_{i=2}^{p} \frac{\hat{e}_{i,q}\nu_{ij}}{(1-\phi_1\phi_i)}, \qquad \textbf{(3.21)}
\end{aligned}$$

where $\pi_1 = 0$ for $v \le q-j-1$, $\pi_1 = 1$ otherwise. Note that, for $j = v = 0$, the first term in equation (3.20) becomes zero. In accordance with the latter, the first four terms in equation (3.21) become zero. After some algebra equation (3.21) gives

$$\begin{aligned}
&\operatorname{cov}(y_t, z_{t-(j+v)}) \\
&= \sum_{i=2}^{p} \frac{\hat{e}_{i1}\lambda_{ij}}{\phi_i - \phi_1} \phi_i^{j+v} + \phi_1^{j+v} \sum_{i=2}^{p} \hat{e}_{i0}\lambda_{ij} \left(\frac{1}{1-\phi_1\phi_i} - \frac{\phi_i}{\phi_i - \phi_1} \right) \\
&\quad - \sum_{i=2}^{p} \hat{e}_{iq}\nu_{ij} \left(\frac{\phi_1^{q+j+v}}{1-\phi_1\phi_i} + \pi_1 \frac{\phi_i^{v+j-q+1} - \phi_1^{v+j-q+1}}{\phi_i - \phi_1} \right) \\
&\quad + \phi_1^{v} \sum_{k=1}^{j}\sum_{i=2}^{p} \hat{e}_{i,j-k} \lambda_{ij,k^-} (\pi_2 \phi_1^k + \phi_1^{2j-k}) + \phi_1^{v} \sum_{k=v+1}^{q-1-j}\sum_{i=2}^{p} \hat{e}_{ij+k} \lambda_{ij,k^+} \phi_1^{2j+k} \\
&\quad + \phi_1^{v} \sum_{k=1}^{\min(v,q-1-j)} \sum_{i=2}^{p} \hat{e}_{i,j+k} \lambda_{ij,k^+} (\phi_1^{-k} + \phi_1^{2j+k}), \qquad \textbf{(3.22)}
\end{aligned}$$

where $\pi_1 = 1$ for $v > q-j-1$, $\pi_1 = 0$, otherwise, and $\pi_2 = 0$ for $k = j$, $\pi_2 = 1$, otherwise.

The jth autocovariance of y_t $(0 \leq j \leq q-1)$ is given by

$$y_{t-j} = \sum_{i=0}^{\infty} \phi_1^i z_{t-j-i} \Rightarrow \text{cov}_j(y_t) = \sum_{i=0}^{\infty} \phi_1^i \,\text{cov}(y_t, z_{t-j-i}). \tag{3.23}$$

Substituting (3.22) into the preceding equation, and after some algebra, we get

$$\text{cov}_j(y_t) = \sum_{i=2}^{p} e_{ij} \lambda_{ij} + \phi_1^j \zeta', \tag{3.24}$$

where

$$\begin{aligned}
\zeta' = {} & \frac{1}{1-\phi_1^2} \sum_{k=1}^{j} \sum_{i=2}^{p} \hat{e}_{i,j-k} \lambda_{ij,k^-} (\pi_1 \phi_1^{k-j} + \phi_1^{j-k}) \\
& + \sum_{v=0}^{q-1-j} \sum_{k=1}^{v} \sum_{i=2}^{p} \phi_1^{2v} \hat{e}_{i,j+k} \lambda_{ij,k^+} (\phi_1^{-k-j} + \phi_1^{j+k}) \\
& + \frac{\phi_1^{2(q-j)}}{1-\phi_1^2} \sum_{k=1}^{q-1-j} \sum_{i=2}^{p} \hat{e}_{i,j+k} \lambda_{ij,k^+} (\phi_1^{-k-j} + \phi_1^{j+k}) \\
& + \sum_{v=0}^{q-2-j} \sum_{k=v+1}^{q-1-j} \sum_{i=2}^{p} \phi_1^{2v} \hat{e}_{i,j+k} \lambda_{ij,k^+} \phi_1^{j+k} \\
& + \frac{1}{1-\phi_1^2} \sum_{i=2}^{p} \hat{e}_{i0} \lambda_{ij} \left(\frac{1}{1-\phi_1 \phi_i} - \frac{\phi_i}{\phi_i - \phi_1} \right) \\
& - \sum_{i=2}^{p} \hat{e}_{iq} \nu_{ij} \left[\frac{\phi_1^q}{(1-\phi_1\phi_i)(1-\phi_1^2)} + \frac{\phi_1^{q-2j}\phi_i}{(\phi_i - \phi_1)(1-\phi_1\phi_i)} \right. \\
& \qquad \left. - \frac{\phi_1^{q-2j+1}}{(\phi_i-\phi_1)(1-\phi_1^2)} \right],
\end{aligned} \tag{3.24a}$$

where $\pi_1 = 1$ for $k = j$, $\pi_1 = 0$ otherwise. We employ the same reasoning with the one used in the first method (p. 631) to get

$$\zeta' = e_{10} \lambda_{1j}. \tag{3.25}$$

Finally, substituting the preceding equation into (3.24) we get (3.6). ■

4. ARMA MODEL REDUNDANCY

COROLLARY 1. *A sufficient condition for the lack of the* ARMA(p,q) *model redundancy is given by*

$$\prod_{i=1}^{p} \lambda_{iq} \neq 0,$$

where λ_{iq} *is given by* (3.6b).

Proof. The proof follows immediately from Theorem 2. ■

A Note on ARMA Model Parameter Redundancy. When the first inverse root of the AR polynomial (ϕ_1) is equal to the first inverse root of the MA polynomial (θ_1^*) the order of the autoregressive part reduces from p to $p-1$. In accordance with the latter, the coefficient of the ϕ_1^j in the jth autocovariance, for $j \geq q$, (λ_{1q}) becomes zero. Moreover, the order of the MA part reduces from q to $q-1$. In accordance with the latter, the autocovariance of order q becomes

$$\mathrm{cov}_q(y_t) = \sum_{i=2}^{p} e_{iq}\lambda_{iq} = \sum_{i=2}^{p} \hat{e}_{i,q}\lambda'_{i,q-1}, \tag{4.1}$$

because

$$\frac{\phi_i \lambda_{iq}}{(1-\theta_1^*\phi_i)(\phi_i - \theta_i^*)} = \lambda'_{i,q-1}, \tag{4.1a}$$

where

$$\lambda'_{i,j} = \sum_{k=0}^{q-1} (\theta'_k)^2 + \sum_{l=1}^{j}\sum_{k=0}^{q-1-l} \theta'_k\theta'_{k+l}(\phi_i^l + \phi_i^{-l})$$
$$+ \sum_{l=j+1}^{q-1}\sum_{k=0}^{q-1-l} \theta'_k\theta'_{k+l}(\phi_i^l + \phi_i^{l-2j}) \tag{4.1b}$$

and

$$\prod_{j=2}^{q}(1-\theta_j^* L) = -\sum_{j=0}^{q-1}\theta'_j L^j, \qquad \theta'_0 = -1 \tag{4.1c}$$

and the autocovariance of order $q-1$ becomes

$$\mathrm{cov}_{q-1}(y_t) = \prod_{i=1}^{p} e_{i,q-1}\lambda_{i,q-1} = \sum_{i=2}^{p} e_{i,q-1}\lambda_{iq} + \sum_{i=1}^{p} e_{i,q-1}\nu_{i,q-1} \tag{4.2}$$

$$= \sum_{i=2}^{p} e_{i,q-1}\lambda_{iq} = \sum_{i=2}^{p} \hat{e}_{i,q-1}\lambda'_{i,q-1}, \tag{4.2a}$$

because

$$\sum_{i=1}^{p} e_{i,q-1}\nu_{i,q-1} = -\theta_q \sum_{i=1}^{p} e_{i,q-1}(\phi_i^{2-q} - \phi_i^q)$$
$$= \theta_q \sum_{i=1}^{p} \frac{\phi_i^{p-2}}{\prod_{(l=1,\, l\neq i)}^{p}(1-\phi_l\phi_i)\prod_{(k=1,\, k\neq i)}^{p}(\phi_i - \phi_k)} = 0. \tag{4.2b}$$

Example.

Let y_t be an ARMA(2,2) process given by

$$y_t = \phi_1 y_{t-1} + z_t, \tag{4.3}$$

where $z_t = \phi_2 z_{t-1} + \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2}$ and where ϕ_1 and ϕ_2 are the two i-roots of the second-order polynomial $1 - \phi_1^* L - \phi_2^* L^2$. The preceding equation says that an ARMA(2,2) process can be expressed as an AR(1) process with an ARMA(1,2) error term (z_t).

The nth autocovariance of y_t (for $n \geq 2$) is given by

$$\begin{aligned} \mathrm{cov}_n(y_t) &= \frac{1}{(1-\phi_1\phi_2)(\phi_1-\phi_2)} \\ &\quad \times \left\{ \frac{\phi_1^n \phi_1[(1+\theta_1^2+\theta_2^2)+(-\theta_1+\theta_1\theta_2)(\phi_1+\phi_1^{-1})-\theta_2(\phi_1^2+\phi_1^{-2})]}{1-\phi_1^2} \right. \\ &\quad \left. - \frac{\phi_2^n \phi_2[(1+\theta_1^2+\theta_2^2)+(-\theta_1+\theta_1\theta_2)(\phi_2+\phi_2^{-1})-\theta_2(\phi_2^2+\phi_2^{-2})]}{1-\phi_2^2} \right\}. \end{aligned} \tag{4.4}$$

The nth autocovariance of y_t has two terms. The first is the product of the first i-root raised to the power of $n(\phi_1^n)$ and a coefficient that depends on ϕ_1, ϕ_2, θ_1, and θ_2. The second term is the product of the second i-root raised to the power of $n(\phi_2^n)$ and a coefficient that also depends on ϕ_1, ϕ_2, θ_1, and θ_2. These two terms are such that if in the one we interchange the two roots we will get the other. And that is in accordance with the fact that the ARMA(2,2) process (4.3) is symmetric with respect to the two i-roots of the second-order AR polynomial. Note also that, if one of the two i-roots ϕ_k, $k = 1,2$ is equal to one of the two i-roots of the MA polynomial $[(1 - \theta_1 L - \theta_2 L^2) = (1 - \theta_1^* L)(1 - \theta_2^* L), -\theta_1^*\theta_2^* = \theta_2, \theta_1^* + \theta_2^* = \theta_1]$ (say $\phi_1 = \theta_1^*$), then the process reduces to an ARMA(1,1) process and the coefficient of ϕ_1^j in the autocovariance function becomes zero.

$$\begin{aligned} \mathrm{cov}_j(y_t) &= \frac{\phi_2^j \phi_2[1+\theta_1^2+\theta_2^2+(-\theta_1+\theta_1\theta_2)(\phi_2+\phi_2^{-1})-\theta_2(\phi_2^2+\phi_2^{-2})]}{(1-\phi_2^2)(1-\phi_1\phi_2)(\phi_2-\phi_1)} \\ &= \frac{\phi_2^j}{1-\phi_2^2}[1+(\theta_2^*)^2-\theta_2^*(\phi_2+\phi_2^{-1})], \quad \text{for } j \geq 1. \end{aligned} \tag{4.5}$$

5. CONCLUSIONS

In this paper we showed that an ARMA(p,q) model can be expressed as a sequence of p AR(1) processes with ARMA$(p-i,q)$ (where $i = 1,2,\dots,p$) error terms. The first AR(1) process has an ARMA$(p-1,q)$ error term, the second has an ARMA$(p-2,q)$ error term, ..., and the last one has a MA(q) error term.

Moreover, we showed that the ARMA(p,q) process is symmetric with respect to the i-roots of the AR polynomial $(\phi_1,\dots,\phi_p)$. Hence, the autocovariance of the process is symmetric with respect to the i-roots of the AR polynomial.

This recursive and symmetric nature of the ARMA(p,q) model led us to find an exact-form solution for the jth autocovariance. We expressed it as an explicit function of the roots of the AR polynomial and the parameters of the MA part. It is the sum of p terms. The ith term is the product of the inverse of the ith root of the AR polynomial, raised to the power of j, times a coefficient that depends on $\phi_1,\dots,\phi_p,\theta_1,\dots,\theta_q$. Because of the aforementioned symmetry, if in the coefficient of ϕ_i^j we interchange ϕ_i with ϕ_k we will get the coefficient of ϕ_k^j.

Furthermore, in our proof the i-roots of the AR polynomial can be either real or complex. The only restriction that we imposed is that all the roots are distinct.

NOTES

1. McLeod (1993) gives a simple condition, expressed in terms of the ARMA model parameters, for determining ARMA model redundancy. He derived the algebraic condition by setting the determinant of the auxiliary matrix of the ARMA model (J) equal to zero.
2. Without loss of generality we assume that the variance of ϵ_t is 1.
3. I am grateful to an anonymous referee from *Econometric Theory* for suggesting this method to me.

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APPENDIX A: CASE OF TWO EQUAL ROOTS (AR(2))

Proof (1st Method). Let y_t be an AR(2) process that is given by

$$y_t = \phi_1 y_{t-1} + z_t, \qquad z_t = \phi_1 z_{t-1} + \epsilon_t. \tag{A.1}$$

The covariance between z_t and y_{t-n} is given by

$$\operatorname{cov}(z_t, y_{t-n}) = \sum_{i=0}^{\infty} \phi_1^i \operatorname{cov}_{n+i}(z_t). \tag{A.2}$$

Because z_t is an AR(1) process, its autocovariance is given by

$$\mathrm{cov}_n(z_t) = \frac{\phi_1^n}{1-\phi_1^2}. \tag{A.3}$$

Substituting (A.3) into (A.2) we get

$$\mathrm{cov}(z_t, y_{t-n}) = \frac{\phi_1^n}{(1-\phi_1^2)^2}. \tag{A.4}$$

From (A.1) we get the variance of y_t,

$$\mathrm{var}(y_t) = \frac{\mathrm{var}(z_t) + 2\phi_1 \,\mathrm{cov}(z_t, y_{t-1})}{1-\phi_1^2}. \tag{A.5}$$

Substituting (A.3) and (A.4) into (A.5) we get

$$\mathrm{var}(y_t) = \frac{1+\phi_1^2}{(1-\phi_1^2)^3}. \tag{A.6}$$

The covariance of y_t is given by

$$\mathrm{cov}_n(y_t) = \phi_1^n \,\mathrm{var}(y_t) + \sum_{i=0}^{n-1} \phi_1^i \,\mathrm{cov}(z_{t-i}, y_{t-n}). \tag{A.7}$$

Substituting (A.4) and (A.6) into the preceding equation we get

$$\mathrm{cov}_n(y_t) = \frac{\phi_1^n}{(1-\phi_1^2)^2}\left(\frac{1+\phi_1^2}{1-\phi_1^2} + n\right). \tag{A.8}$$

■

Proof (2nd Method).[3] Taking the limit of (2.2) as $(\phi_2 \to \phi_1)$ we get

$$\begin{aligned}
\mathrm{cov}_n(y_t) &= \lim_{\phi_2\to\phi_1} \frac{1}{(1-\phi_1\phi_2)(\phi_1-\phi_2)}\left(\frac{\phi_1^n\phi_1}{1-\phi_1^2} - \frac{\phi_2^n\phi_2}{1-\phi_2^2}\right)\\
&= \lim_{\phi_2\to\phi_1} \frac{1}{(1-\phi_1\phi_2)(1-\phi_1^2)(1-\phi_2^2)} \lim_{\phi_2\to\phi_1} \frac{\phi_1^n\phi_1(1-\phi_2^2) - \phi_2^n\phi_2(1-\phi_1^2)}{\phi_1-\phi_2}\\
&= \frac{1}{(1-\phi_1^2)^3} \lim_{\phi_2\to\phi_1} \frac{\dfrac{\partial}{\partial\phi_2}[\phi_1^n\phi_1(1-\phi_2^2) - \phi_2^n\phi_2(1-\phi_1^2)]}{\dfrac{\partial}{\partial\phi_2}(\phi_1-\phi_2)}\\
&= \frac{1}{(1-\phi_1^2)^3} \lim_{\phi_2\to\phi_1} \frac{[\phi_1^n\phi_1(-2\phi_2) - (n+1)\phi_2^n(1-\phi_1^2)]}{-1}\\
&= \frac{\phi_1^n}{(1-\phi_1^2)^2}\left(\frac{1+\phi_1^2}{1-\phi_1^2} + n\right).
\end{aligned} \tag{A.9}$$

This method will be applicable to all multiple roots cases. ■

APPENDIX B: CASE OF THREE EQUAL ROOTS (AR(3))

Proof. Let y_t be an AR(3) process that we express as a sequence of three AR(1) processes:

$$y_t = \phi_1 y_{t-1} + z_t, \qquad z_t = \phi_1 z_{t-1} + x_t, \qquad x_t = \phi_1 x_{t-1} + \epsilon_t. \tag{B.1}$$

Substituting (A.8) into (A.2) we get

$$\operatorname{cov}(z_t, y_{t-n}) = \frac{\phi_1^n}{(1-\phi_1^2)^3}\left(\frac{1+2\phi_1^2}{1-\phi_1^2} + n\right). \tag{B.2}$$

Substituting (A.8) and (B.2) into (A.5) we get the variance of y_t,

$$\operatorname{var}(y_t) = \frac{1+4\phi_1^2+\phi_1^4}{(1-\phi_1^2)^5}. \tag{B.3}$$

Substituting equations (B.3) and (B.2) into (A.7) we get the nth autocovariance of y_t,

$$\operatorname{cov}_n(y_t) = \frac{\phi_1^n}{(1-\phi_1^2)^3}\left[\frac{1+4\phi_1^2+\phi_1^4}{(1-\phi_1^2)^2} + \frac{1+\phi_1^2}{1-\phi_1^2}\left(\frac{3n}{2}\right) + \frac{n^2}{2}\right]. \tag{B.4}$$

■

APPENDIX C: CASE OF FOUR EQUAL ROOTS (AR(4))

Proof. Let y_t be an AR(4) process that we express as a sequence of four AR(1) processes:

$$y_t = \phi_1 y_{t-1} + z_t, \qquad z_t = \phi_1 z_{t-1} + d_t, \qquad d_t = \phi_1 d_{t-1} + x_t, \qquad x_t = \phi_1 x_{t-1} + \epsilon_t. \tag{C.1}$$

Substituting equation (B.4) into (A.2) we get the covariance between z_t and y_{t-n},

$$\operatorname{cov}(z_t, y_{t-n}) = \frac{\phi_1^n}{(1-\phi_1^2)^4}\left[\frac{1+6\phi_1^2+3\phi_1^4}{(1-\phi_1^2)^2} + \frac{3+5\phi_1^2}{1-\phi_1^2}\left(\frac{n}{2}\right) + \frac{n^2}{2}\right]. \tag{C.2}$$

Substituting equations (C.2) and (B.3) into (A.5) we get the variance of y_t,

$$\operatorname{var}(y_t) = \frac{1+9\phi_1^2+9\phi_1^4+\phi_1^6}{(1-\phi_1^2)^7}. \tag{C.3}$$

Substituting equations (C.2) and (C.3) into (A.7) we get the nth autocovariance of y_t,

$$\operatorname{cov}_n(y_t) = \frac{\phi_1^n}{(1-\phi_1^2)^4}\left[\frac{1+9\phi_1^2+9\phi_1^4+\phi_1^6}{(1-\phi_1^2)^3} + \frac{(11+38\phi_1^2+11\phi_1^4)n}{6(1-\phi_1^2)^2} + \frac{1+\phi_1^2}{1-\phi_1^2}n^2 + \frac{n^3}{6}\right] \qquad \text{(C.4)}$$

■