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The impulse response function of the long memory GARCH process

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Abstract

In this article we derive convenient representations for the cumulative impulse response function of the long memory GARCH(p, d, q) (LMGARCH) process. Our results extend the results in Baillie et al. (1996) [Baillie, R.T., Bollerslev, T., Mikkelsen, H.O. 1996. Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 74, 3–30.] on the first order LMGARCH. Using the derived impulse response functions we compare the persistence of shocks to the conditional variance in various GARCH models of interest such as stable, integrated and LMGARCH.

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1. Introduction

The topic of long memory and persistence has recently attracted considerable attention in terms of the second moment of a process. An excellent survey of major econometric work on long memory processes and their applications in economics and finance is given by Baillie (1996). The issue of temporal dependence on financial time series has been the focus of attention since information on persistence can also guide the search for an economic explanation of movements in asset returns. For example, as Baillie et al. (1996) point out, there is a direct relation between the long term dependence in the conditional

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variances of daily spot exchange rates and the long memory in the forward premium. This relation could explain the systematic rejection of the unbiasedness hypothesis as an artefact due to the unbalanced regression of the return on the premium (Baillie et al., 1996).

Robinson (1991) was the first to consider the long memory potential of a model which he called linear ARCH (LARCH). Subsequently, many researchers have proposed extensions of GARCH-type models which can produce long memory behavior (see, for example, Teysnière, 1998; Davidson, 2004 and Giraitis et al., 2004, and the references therein). The empirical relevance of long memory conditional heteroscedasticity has emerged in a variety of studies of time series of economic and financial variables (see, for example, Conrad and Karanasos, 2005a,b). Kirman and Teysnière (2005) assemble models from economic theory providing plausible micro foundations for the occurrence of long memory in economics. Recent research has been aimed at both extending our understanding of these well established models, and widening the range of data features that can be handled. For example, Giraitis et al. (2005) provide an overview of recent theoretical findings on the LMGARCH processes.

Moreover, Baillie et al. (1996) measure the persistence of shocks to the conditional variance using impulse response functions (IRF). To appreciate how such measures work in practice they consider the fractionally integrated GARCH (FIGARCH) process of order $(1, d, 0)$. To that end, in this article the impulse response function is analyzed in the framework of an LMGARCH(p, d, q) process. We also look at the persistence of shocks in the conditional variance process for the LMGARCH model as compared with the persistence of shocks for the stable and integrated GARCH models. An important related study by Karanasos et al. (2004) derives convenient representations for the autocorrelation function of the squared values of LMGARCH processes, and some of our results can be seen as complementary to theirs.

This article is organized as follows. Section 2 lays out the model of interest, assumptions and notation. Section 3 presents expressions for the cumulative impulse response functions of the LMGARCH(p, d, q) process and discusses an empirical example. In the conclusions we suggest future developments.

2. The long memory GARCH model

To establish terminology and notation, recall from Karanasos et al. (2004) that an LMGARCH(p, d, q) process $\{\varepsilon_t\}$ is defined by the equation

$$\varepsilon_t = e_t \sqrt{h_t}, \quad t \in \mathbb{Z}, \quad (1)$$

and

$$h_t = \omega + [\Omega(L) - 1]v_t, \quad (2)$$

with

$$\Omega(L) = \sum_{j=0}^{\infty} \omega_j L^j \triangleq \frac{B(L)}{\Phi(L)(1-L)^d},$$

where $v_t \triangleq \varepsilon_t^2 - h_t$. Here and in the remainder of this article, L stands for the lag operator and the symbol ‘ \triangleq ’ is used to indicate equality by definition. We assume hereafter that $\omega \in (0, \infty)$, $0 < d < 0.5$ and that the finite order polynomials $\Phi(L) \triangleq 1 - \sum_{i=1}^q \phi_i L^i = \prod_{i=1}^q (1 - \zeta_i L)$, $B(L) \triangleq - \sum_{i=1}^p \beta_i L^i (\beta_0 \triangleq -1)$ have zeros outside the unit circle in the complex plane.

The rescaled innovations e_t are assumed to be *i.i.d* with $\mathbb{E}(e_t) = \mathbb{E}(e_t^2 - 1) = 0$. By Eq. (1) and the *i.i.d.*-ness of the e_t , $\mathbb{E}(v_t | \mathcal{F}_{t-1}) = 0$ where \mathcal{F}_t is the σ -field of events induced by $\{\varepsilon_s, s \leq t\}$. For notational convenience, in what follows we denote $\varpi_m \triangleq \sum_{j=0}^{\infty} \omega_j \omega_{j+m}$ ($m=0, 1, 2, \dots$). Note that, since $d < 0.5$, $\varpi_0 < \infty$. The e_t^2 has finite first moment equal to ω . In addition, simple manipulations suggest that $\mathbb{E}(e_t^4) < \infty$ and $[\mathbb{E}(e_t^4) - 1] \varpi_0 < \mathbb{E}(e_t^4)$ imply covariance stationarity of the e_t^2 . Under these conditions the autocorrelations $\{\rho_m(e_t^2) \triangleq \text{Corr}(e_{t+m}^2, e_t^2)\}$ are $\rho_m(e_t^2) = \varpi_m / \varpi_0$ (see Karanasos et al., 2004).

Moreover, h_t has an ARCH(∞) representation, i.e. it can be expressed as an infinite distributed lag of e_{t-j}^2 terms as $(h_t - \omega) = \Psi(L)(e_t^2 - \omega)$ where $\Psi(L) = \sum_{j=1}^{\infty} \psi_j L^j \triangleq [1 - \Phi(L)(1 - L)^d / B(L)]$. Note that in this specification the conditional variance and the squared errors are expressed in deviations from the unconditional variance ω . To guarantee the non-negativity of the conditional variance one has to impose inequality constraints which ensure that $\psi_j \geq 0$ for $j=1, 2, \dots$. Necessary and sufficient constraints for $p \leq 2$ and sufficient constraints for $p > 2$ can be found in Conrad and Haag (2005).

Furthermore, under Eqs. (1) and (2), the coefficients ω_j decay at a slow hyperbolic rate so that $\omega_j = O(j^{d-1})$. This in turn implies that the autocorrelations satisfy $\rho_m(e_t^2) = O(m^{2d-1})$, provided $\mathbb{E}(e_t^4) < \infty$. Hence, when the fourth moment of the ε_t exists, e_t^2 is a weakly stationary process which exhibits long memory for all $d \in (0, 0.5)$, in the sense that the series $\sum_{m=0}^{\infty} |\rho_m(e_t^2)|$ is properly divergent. For this reason, we refer to a process ε_t satisfying Eqs. (1) and (2) as an LMGARCH(p, d, q) process.

Finally, it is interesting to note the difference between the LMGARCH process and the FIGARCH model. The ARCH(∞) formulation of the latter is $h_t = \omega + \Psi(L)e_t^2$. It is noteworthy that this model, unless $\mathbb{E}(e_t^2 - 1) < 0$, is not compatible with covariance stationary ε_t . However, Zaffaroni (2004) points out that even this weak covariance stationarity condition for the levels ε_t rules out long memory in the squares e_t^2 .

3. The IRF of the LMGARCH(p, d, q) model

In the LMGARCH class of models, the short-run behavior of the time series is captured by the conventional ARCH and GARCH parameters, while the long-run dependence is conveniently modelled through the fractional differencing parameter.

Since in the LMGARCH the conditional variance is parameterized as a linear function of the past squared innovations, the persistence of the conditional variance is most simply characterized in terms of the impulse response coefficients for the optimal forecast of the future conditional variance as a function of the current innovation v_t . Following Baillie et al. (1996) we define the impulse response and cumulative impulse response function as follows:

Definition 1. The impulse response function of the LMGARCH(p, d, q) model is given by the sequence δ_k , $k=0, 1, \dots$, where $\delta_k \triangleq \partial \mathbb{E}_t(e_{t+k}^2) / \partial v_t - \partial \mathbb{E}_t(e_{t+k-1}^2) / \partial v_t$, with $\delta_0 \triangleq 1$. Accordingly, the cumulative impulse response function is given by the sequence $\lambda_k \triangleq \sum_{l=0}^k \delta_l$.

The impulse response coefficients δ_k can be simply obtained by considering the first difference of the squared errors

$$(1 - L)e_t^2 = \Delta(L)v_t, \quad (3)$$

where $\Delta(L) = \sum_{j=0}^{\infty} \delta_j L^j = (1 - L)\Omega(L)$. Since by definition the impulse response coefficients δ_k are related to the cumulative impulse response weights λ_k by $\Delta(L) = (1 - L)A(L)$, with $A(L) = \sum_{k=0}^{\infty} \lambda_k L^k$, it follows that the cumulative impulse response weights λ_k coincide with the ω_k coefficients defined by Eq. (2).¹

Further, let F be the Gaussian hypergeometric function defined by $F(a, b; c; z) \triangleq \sum_{j=0}^{\infty} \frac{(a)_j (b)_j z^j}{(c)_j j!}$, where $(a)_j \triangleq \prod_{i=0}^{j-1} (a + i)$ (with $(a)_0 = 1$) is Pochhammer's shifted factorial. Then, the fractional differencing operator $(1 - L)^d$ in Eq. (2) is most conveniently expressed in terms of the hypergeometric function

$$(1 - L)^d = F(-d, 1; 1; L) = \sum_{j=0}^{\infty} \frac{\Gamma(j - d)}{\Gamma(j + 1)\Gamma(-d)} L^j \triangleq \sum_{j=0}^{\infty} g_{-d}(j) L^j,$$

where $\Gamma(\cdot)$ is the gamma function. Note that using this notation we can write $\Delta(L) = \frac{B(L)}{\Phi(L)} F(d - 1, 1; 1; L)$.

Next, we establish a representation for the cumulative impulse response function of the LMGARCH(p, d, q) process.

Theorem 1. *The cumulative impulse response function $\lambda_k, k = 0, 1, \dots$, of the LMGARCH (p, d, q) model is given by*

$$\lambda_k = \left(1 - \sum_{i=1}^{\min\{k,p\}} \beta_i \right) \sum_{i=0}^{\max\{k-p,0\}} \zeta_i g_d(\max\{k - p, 0\} - i) + \sum_{i=\max\{k-p,0\}+1}^k \left(1 - \sum_{l=1}^{k-i} \beta_l \right) c_i,$$

where

$$c_i \triangleq \sum_{l=0}^i \zeta_{i-l} g_{d-1}(l), \text{ with } \zeta_i \triangleq \sum_{l=1}^q \theta_l \zeta_l^i, \text{ and } \theta_i \triangleq \frac{\zeta_i^{q-1}}{\prod_{l=1, l \neq i}^q (\zeta_i - \zeta_l)}. \quad (4)$$

(Recall that the ζ_i are defined by Eq. (2)).

Proof. In view of Eq. (3), we have

$$\Delta(L) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} -c_{j-i} \beta_i L^j, \text{ or } \delta_j = \sum_{i=0}^p -c_{j-i} \beta_i,$$

where c_i is defined in Theorem 1 and $c_{-i} = 0 (i = 1, \dots, p)$.

Therefore, the cumulative impulse response function is given by

$$\lambda_k = \sum_{i=0}^k \delta_i = \left(- \sum_{i=0}^{\min\{k,p\}} \beta_i \right) \sum_{i=0}^{\max\{k-p,0\}} c_i + \sum_{i=\max\{k-p,0\}+1}^k \left(1 - \sum_{l=1}^{k-i} \beta_l \right) c_i,$$

where we use the convention that $\sum_{j=1}^0 \beta_j = 0$.

¹ Note that since Eq. (3) is satisfied by both the LMGARCH and the FIGARCH process the results that follow can be applied to the latter as well.

Hence, in view of the fact that

$$\sum_{i=0}^{\max\{k-p,0\}} c_i = \sum_{i=0}^{\max\{k-p,0\}} \zeta_i g_d(\max\{k-p,0\} - i),$$

Eq. (4) follows. \square

The long-run impact of past shocks for the volatility process may now be assessed in terms of the limit of the cumulative impulse response weights, i.e., $\Delta(1) = \lim_{k \rightarrow \infty} \sum_{l=0}^k \delta_l = \lim_{k \rightarrow \infty} \lambda_k$. Note that the results in Theorem 1 hold for $0 \leq d \leq 1$.

As noted by Baillie et al. (1996), for $0 \leq d < 1$, $F(d-1, 1; 1; 1) = 0$, so that for the LMGARCH(p, d, q) model with $0 < d < 0.5$ and the stable GARCH model with $d = 0$, shocks to the conditional variance will ultimately die out in a forecasting sense. In contrast, for $d = 1$, $F(d-1, 1; 1; 1) = 1$, and the cumulative impulse response weights will converge to the nonzero constant $\Delta(1) = B(1)/\Phi(1)$. Thus, from a forecasting perspective shocks to the conditional variance of the integrated GARCH (IGARCH) model persist indefinitely.

To illustrate the general result we consider the LMGARCH(1, $d, 1$) process. In this case $B(L) = 1 - \beta_1 L$ and $\Phi(L) = 1 - \phi_1 L$.

Lemma 1. *The cumulative impulse response coefficients λ_k of the LMGARCH(1, $d, 1$) are given by*

$$\lambda_k = g_d(k) + (\phi_1 - \beta_1) \sum_{i=1}^k \phi_1^{i-1} g_d(k-i). \quad (5)$$

Moreover, when $\phi_1 = 0$, Eq. (5) gives the cumulative impulse response weights of the LMGARCH(1, $d, 0$) model: $\lambda_k = [1 - \beta_1 - (1-d)/k] \cdot g_d(k-1)$.² By restricting d to being zero in Eq. (5) and observing that $g_0(0) = 1$ and $g_0(j) = 0$ for $j > 0$, we obtain the cumulative impulse response function of the GARCH(1, 1) model: $\lambda_k = (\phi_1 - \beta_1) \phi_1^{k-1}$. Finally, when $d = 0$ and $\phi_1 = 1$, we obtain the λ_k of the IGARCH(1, 1) model: $\lambda_k = (1 - \beta_1)$.

The impulse response functions can be used:

- (a) to distinguish between short and long memory specifications. Conrad and Karanasos (2005a) model the conditional variance of the monthly US inflation rate for the period 1962–2000 as GARCH(1, 1), IGARCH(1, 1) and FIGARCH(1, $d, 1$), respectively. Fig. 1 plots the cumulative impulse response functions of their parameter estimates for the GARCH(1, 1) model with $\hat{\phi}_1 = 0.976$ and $\hat{\beta}_1 = 0.822$, the IGARCH(1, 1) process with $\hat{\beta}_1 = 0.819$ and the FIGARCH(1, $d, 1$) specification with $\hat{\phi}_1 = 0.325$, $\hat{\beta}_1 = 0.768$ and $\hat{d} = 0.692$. Note that while a shock to the optimal forecast of the future conditional variance decays at an exponential rate in the GARCH model, and remains important for forecasts of all horizons in the IGARCH model, in sharp contrast the effect will die out at a slow hyperbolic rate in the FIGARCH model.

² The cumulative impulse response function of the LMGARCH(1, $d, 0$) model was first derived by Baillie et al. (1996) (see also, Eq. (87) in Baillie, 1996).

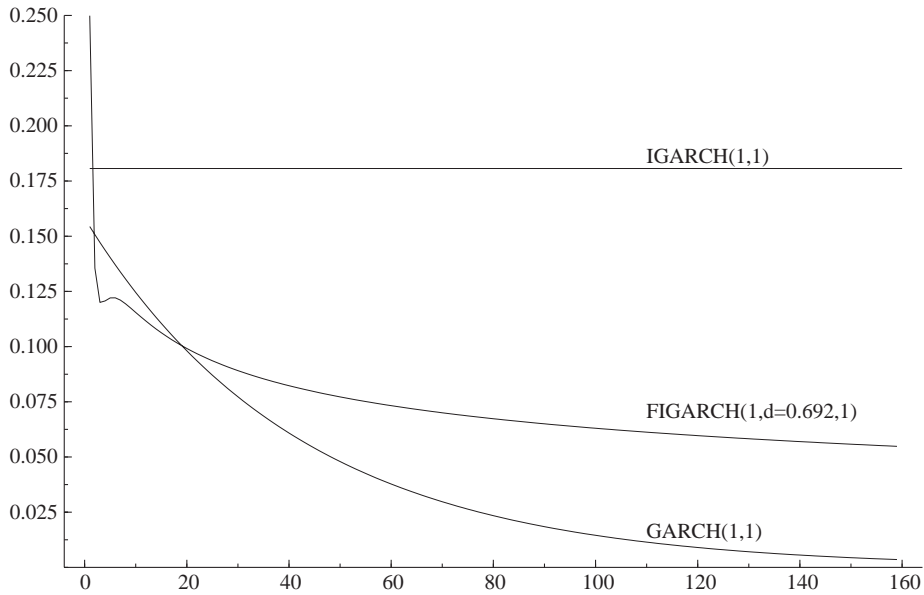


Fig. 1. Cumulative impulse response functions for GARCH, IGARCH and FIGARCH models.

(b) to investigate the persistence properties of a particular LMGARCH specification for different parameter values. For example, [Conrad and Haag \(2005\)](#) show that in the LMGARCH(1, d , 0) model one can allow for $\beta_1 < 0$ (there is a lower bound for β_1 depending on the value of d). [Fig. 2](#) plots the impulse response function for the LMGARCH(1, d , 0) for d fixed at 0.45 and with

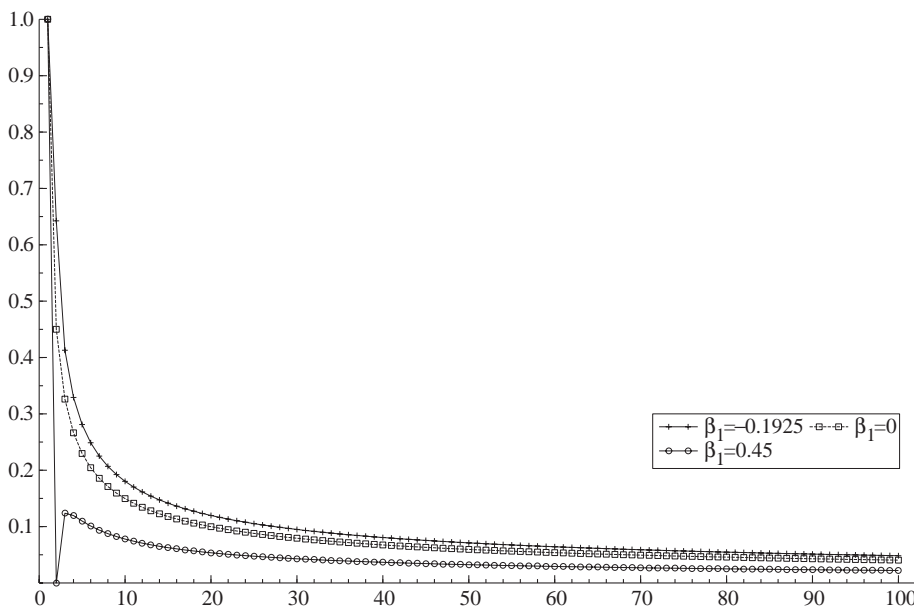


Fig. 2. Cumulative impulse response functions for LMGARCH(1, d , 0) with $d=0.45$ and $\beta_1 \in \{-0.1925, 0, 0.45\}$.

Table 1
Quasi-maximum likelihood estimates

	LMGARCH(1, d , 0)	LMGARCH(0, d , 1)	LMGARCH(1, d , 1)
\hat{d}	0.2326 (0.0365)	0.1847 (0.0237)	0.3805 (0.0680)
$\hat{\phi}_1$	–	–0.1260 (0.0306)	0.2742 (0.0471)
$\hat{\beta}_1$	0.1973 (0.0460)	–	0.6114 (0.0620)

Figures in parentheses are asymptotic standard errors.

The parameter estimates correspond to those in Karanasos et al. (2004), p. 278.

$\beta \in \{-0.1925, 0, 0.45\}$, which is the range of β_1 values allowed for by Proposition 1 in Conrad and Haag (2005). Clearly, the impulse response functions for the three different values of β_1 help to show that persistence is decreasing in β_1 .

3.1. Illustrative example

As an empirical illustration, we examine the properties of the continuously compounded daily rate of returns for the Deutschmark exchange rate vis-a-vis the US dollar over the period from 31/10/1983 to 31/12/1992 (2394 observations in total). This data set was used by Karanasos et al. (2004). We compare the cumulative IRF of LMGARCH(1, d , 0), LMGARCH(0, d , 1) and LMGARCH(1, d , 1) models. The cumulative impulse response weights were evaluated using the formula in Eq. (5) and the quasi-maximum likelihood parameter estimates, reported in Table 1, obtained under the assumption of conditional Gaussianity.

Fig. 3 plots the cumulative IRF for lags up to 160. The cumulative impulse response weights of the LMGARCH(1, d , 0) and LMGARCH(0, d , 1) decrease much faster than that of the LMGARCH(1, d , 1). The plots for the autocorrelation functions in Karanasos et al. (2004) show a very similar pattern.

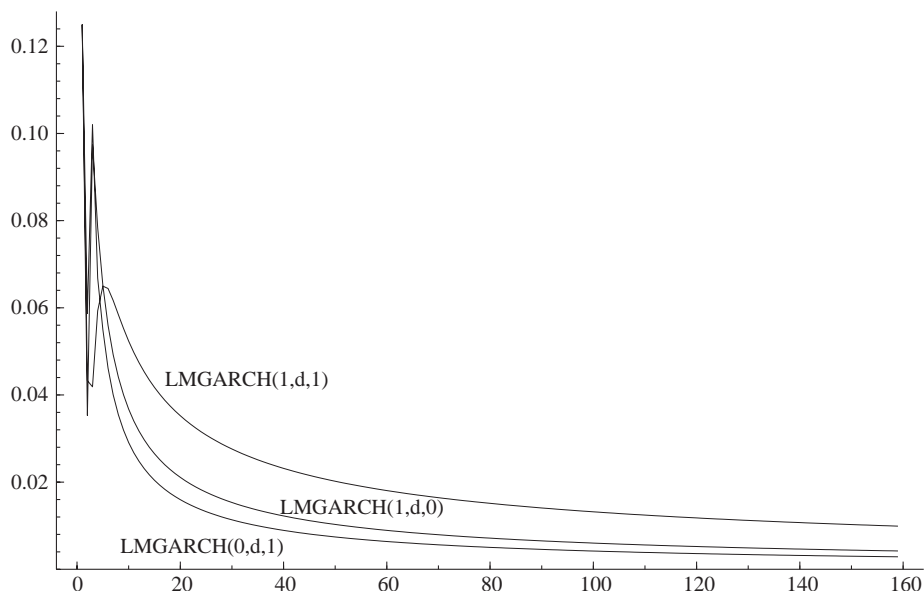


Fig. 3. Cumulative impulse response functions for LMGARCH models from Table 1.

4. Conclusions

In this article we have obtained convenient representations for the cumulative impulse response weights of a process with long memory conditional heteroscedasticity. Since the long memory GARCH model includes the stable and integrated GARCH models as special cases our theoretical results provide a useful tool which facilitates comparison between these major classes of GARCH models. It is worth noting that our results on the impulse response function of the general LMGARCH(p, d, q) model extend the results in Baillie et al. (1996) on the first order LMGARCH model. We should also mention that the methodology used in this article can be applied to obtain the impulse response weights of more sophisticated long memory GARCH models, e.g. ARFIMA, asymmetric power, and multivariate LMGARCH models.

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References

- Baillie, R.T., 1996. Long memory processes and fractional integration in econometrics. *Journal of Econometrics* 73, 5–59.
- Baillie, R.T., Bollerslev, T., Mikkelsen, H.O., 1996. Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 74, 3–30.
- Conrad, C., Haag, B.R. 2005. Inequality Constraints in the Fractionally Integrated GARCH Model. Unpublished Manuscript, University of Mannheim.
- Conrad, C., Karanasos, M., 2005a. On the inflation-uncertainty hypothesis in the USA, Japan and the UK: a dual long memory approach. *Japan and the World Economy* 17, 327–343.
- Conrad, C., Karanasos, M., 2005b. Dual Long Memory in Inflation Dynamics Across Countries of the Euro Area and the Link Between Inflation Uncertainty and Macroeconomic Performance. Unpublished Manuscript, University of Newcastle.
- Davidson, J., 2004. Moment and memory properties of linear conditional heteroscedasticity models, and a new model. *Journal of Business and Economic Statistics* 22, 16–19.
- Giraitis, L., Robinson, P.M., Surgailis, D., 2004. LARCH, leverage and long memory. *Journal of Financial Econometrics* 2, 177–210.
- Giraitis, L., Leipus, R., Surgailis, D., 2005. Recent advances in ARCH modelling. In: Kirman, A., Teyssi re, G. (Eds.), *Long-Memory in Economics*. Springer, Berlin.
- Karanasos, M., Psaradakis, Z., Sola, M., 2004. On the autocorrelation properties of long-memory GARCH processes. *Journal of Time Series Analysis* 25, 265–281.
- Kirman, A., Teyssi re, G. (Eds.), 2005. *Long-Memory in Economics*. Springer, Berlin.
- Robinson, P.M., 1991. Testing for strong serial correlation and dynamic conditional heteroscedasticity in multiple regression. *Journal of Econometrics* 47, 67–84.
- Teyssi re, G., 1998. Multivariate long-memory ARCH modelling for high frequency foreign exchange rates. Proceedings of the Second High Frequency Data in Finance (HFDF-II) Conference. Olsen & Associates, Z rich. April 1998.
- Zaffaroni, P., 2004. Stationarity and memory of ARCH(∞) models. *Econometric Theory* 20, 147–160.