Business School, Brunel University
MSc. EC5501/5509 Modelling Financial Decisions and Markets/Introduction to Quantitative Methods
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Lecture Notes 7

1. Dynamic Regression Models

In time series models, a substantial period of time may pass between the economic decision-making period and the final impact of a change in a policy variable. One can say that it is the nature of economic relationships that the adjustment of y to changes in x is distributed widely through time. If the appropriate decision-and-response period is sufficiently long, lagged explanatory variables should be included explicitly in the model.

One way to model the dynamic responses is to include lagged values of x on the right hand side of the regression equation; this is the basis of the distributed-lag model, in which a series of lagged explanatory variables accounts for the time adjustment process. The (finite) **distributed-lag** model takes the form:

$$y_t = \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + ... + \beta_k x_{t-k} + \varepsilon_t.$$

Given the above model, if the explanatory (input) variable x undergoes a *one-off unit change (impulse)* in some period t, then the immediate impact on y is given by β_0; β_1 is the impact on y after one period, β_2 is the impact after two periods, and so on. The final impact on y is β_k and it occurs after k periods. So speaking, it takes k periods for the full effects of the impulse to be realised. The sequence of coefficients $\{\beta_0, \beta_1, \beta_2, ..., \beta_k\}$ constitutes the *impulse response function* of the mapping from x_t to y_t.

Furthermore, the dynamic behaviour of an economy can reveal itself through a dependence of the current value of an economic variable on its own past values. Specifically, models of how decision makers' expectations are formed, and how they react to changes in the economy, result in the value of y_t depending on lagged y's. So, an additional/alternative way to capture the dynamic component of economic behaviour is to include lagged values of the dependent variable on the right-hand side of the regression together with the exogenous variable.

In time-series econometric modeling a dynamic regression will usually include both lagged dependent and independent variables as regressors:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + \beta_0 x_t + \beta_1 x_{t-1} + ... + \beta_k x_{t-k} + \varepsilon_t.$$

The above model is called the **autoregressive distributed-lag** model, abbreviated as **ARDL**(p, k). The values of p and k (i.e how many lags of y and x will be used) are chosen (i) on the basis of the statistical significance of the lagged variables, and (ii) so that the resulting model is well specified (e.g. it does not suffer from serial correlation).

1.1. The Lag and Difference Operators

In examining dynamics it is useful to use the *lag operator* (L), also known as the backward shift (B) operator. The lag operator can be manipulated in a similar way to any other algebraic quantity:

$$\begin{aligned}
Ly_t &= y_{t-1},\ L^2 y_t = y_{t-2},\ L^3 y_t = y_{t-3}, ..., L^p y_t = y_{t-p}; \\
y_t - y_{t-1} &= (1 - L)\, y_t,\ y_t - y_{t-1} - y_{t-2} = \left(1 - L - L^2\right) y_t; \\
y_t + \alpha y_{t-1} + \alpha^2 y_{t-2} + ... + \alpha^p y_{t-p} &= \left(1 + \alpha L + \alpha^2 L^2 + ... + \alpha^p L^p\right) y_t; \\
\left(1 + \alpha L + \alpha^2 L^2 + \alpha^3 L^3 ....\right) &= \frac{1}{1 - \alpha L},\ \text{if } |\alpha| < 1.
\end{aligned}$$

Another operator which is commonly used in dynamic analysis is the first *difference operator* (Δ), given below:

$$\begin{aligned}
\Delta &\equiv 1 - L,\ \text{i.e. } \Delta y_t = (1 - L)\, y_t = y_t - y_{t-1}; \\
\Delta^2 &\equiv (1 - L)^2,\ \text{i.e. } \Delta^2 y_t = (1 - L)^2\, y_t = \Delta y_t - \Delta y_{t-1}; \\
\Delta^k &\equiv (1 - L)^k,\ k = 1, 2, 3, ...
\end{aligned}$$

1.2. The first order autoregressive distributed-lag model: ARDL(1,1)

The ARDL(1,1), or alternatively the first order *Dynamic Linear Regression Model*, takes the form:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t,\ t = 1, 2, ..., T. \tag{1}$$

Note that y_t is *stable* (i.e. it will converge to its equilibrium) if $-1 < \alpha < 1$.

- If the above stability condition is satisfied, the *long-run solution* (or steady state) of eq. (1) is given by

$$\begin{aligned} y_t &= \frac{\alpha_0}{1-\alpha_1} + \left(\frac{\beta_0+\beta_1}{1-\alpha_1}\right) x_t + \frac{\varepsilon_t}{1-\alpha_1} \\ &= c_0 + c_1 x_t + \frac{\varepsilon_t}{1-\alpha_1}. \end{aligned} \tag{2}$$

The above equation (2) is obtained by assuming that in equilibrium $y_t = y_{t-1}$, and $x_t = x_{t-1}$. So speaking, the *long-run equilibrium* or *target value* for y at time t is given by

$$y_t^* = c_0 + c_1 x_t. \tag{2'}$$

Calculating the long-run solution is a way of recovering the information about the equilibrium provided by the dynamic model (1).[1] One should adopt the static model (2) if agents are always at their target value except for a random error. If, on the other hand, y_t is not in equilibrium then the appropriate specification is a dynamic one.

- The dynamic linear regression model (1) can be *reparameterised* as:

$$\Delta y_t = d_0 + d_1 y_{t-1} + d_2 \Delta x_t + d_3 x_{t-1} + \varepsilon_t, \tag{3}$$

where $d_0 = \alpha_0$, $d_1 = \alpha_1 - 1$, $d_2 = \beta_0$, $d_3 = \beta_0 + \beta_1$.[2] Observe that eq. (1) and (3) are alternative representations of the same statistical model; they only differ in their economic interpretations. In other words, model (1) cannot be tested versus model (3). Also note that in both eq. (1) and (3) we are estimating the same number of parameters. Since eq. (1) and (3) represent the same statistical model, their residual series, sum of squared residuals, and standard errors of the regression will be identical.[3]

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[1] If the variables are in logs then c_1 is the long-run elasticity of y with respect to x.

[2] The relationship between the parameters of eq. (1) and (3) can be obtained by writing (3) as:

$$\begin{aligned} y_t - y_{t-1} &= d_0 + d_1 y_{t-1} + d_2 x_t - d_2 x_{t-1} + d_3 x_{t-1} + \varepsilon_t \Rightarrow \\ y_t &= d_0 + (1+d_1) y_{t-1} + d_2 x_t + (d_3 - d_2) x_{t-1} + \varepsilon_t. \end{aligned}$$

[3] One way to give an economic motivation for the dynamic eq. (3) or (1) is to assume that agents adjust both to the change in the target value (Δy_t^*) and the deviation from target in the

- Let us examine now the case where x_t is added to the right-hand side (RHS) of eq. (3):

$$\Delta y_t = d_0 + d_1 y_{t-1} + d_2 \Delta x_t + d_3 x_{t-1} + d_4 x_t + \varepsilon_t. \tag{4}$$

The regression model (4) cannot be estimated due to the problem of **perfect multicollinearity.** Note that the RHS variable Δx_t is a linear combination of the RHS variables x_t and x_{t-1} : $\Delta x_t = x_t - x_{t-1}$. In other words, the matrix of explanatory variables does not have full rank (not all of its columns/vectors are linearly independent).

- Next, suppose that in the context of the dynamic linear regression model (1) we wish to test the joint statistical significance of the lagged variables. The null and alternative hypotheses are:

$$\begin{aligned} H_0 &: \alpha_1 = 0 \text{ and } \beta_1 = 0; \\ H_1 &: \alpha_1 \neq 0 \text{ and } \beta_1 \neq 0. \end{aligned}$$

When we impose the above restrictions on the (unrestricted) dynamic model (1) we get the static model:

$$y_t = a + b x_t + u_t. \tag{5}$$

We can test the <u>restricted</u> model (5) against the <u>unrestricted</u> model (1) by using either the $F - test$:

$$F - test = \frac{\left(\sum_t \widehat{u}_t^2 - \sum_t \widehat{\varepsilon}_t^2\right)/2}{\left(\sum_t \widehat{\varepsilon}_t^2\right)/(T-4)} \sim F(2, T-4),$$

or the asymptotic *Likelihood Ratio test* ($LR - test$) computed as:

$$LR - test = 2\left(MLL_{UR} - MLL_R\right) \sim \chi^2(r),$$

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previous period $\left(y_{t-1} - y_{t-1}^*\right)$:

$$\Delta y_t = \lambda_1 \Delta y_t^* - \lambda_2 \left(y_{t-1} - y_{t-1}^*\right) + \varepsilon_t,$$

where y_t^* is given by eq. (2').

where MLL_{UR} denotes the maximum log-likelihood of the unrestricted model (regression (1) in this example), MLL_R denotes the maximum log-likelihood of the restricted model (regression (5) in this example), and r denotes the number of restrictions (two in this example). Rejection of the restrictions means that the dynamic linear regression model fits the data better than the static one. When testing one model versus another note that: (i) the restrictions are always expressed in terms of the parameters of the unrestricted model, (ii) in the restricted model we always have to estimate fewer parameters than in the unrestricted one, and (iii) an $F - test$ *can be used only when the restrictions are linear*; non-linear restrictions can be tested using an $LR - test$ or some other asymptotic testing procedure (e.g. an $LM - test$).

- Finally, in the context of model (3), suppose that we are interested in testing the restriction:

$$\begin{aligned} H_0 &: d_1 = -d_3; \\ H_1 &: d_1 \neq -d_3. \end{aligned}$$

This restriction implies that the long-run coefficient of x is unity. When we impose the above restriction on (3) we obtain the following model:

$$\Delta y_t = b_0 + b_1 \Delta x_t + b_2 (y_{t-1} - x_{t-1}) + u_t. \quad (6)$$

We can test the restricted model (6) against the unrestricted model (3) by using either an $F - test$:

$$F - test = \frac{\left(\sum_t \widehat{u}_t^2 - \sum_t \widehat{\varepsilon}_t^2 \right) /1}{\left(\sum_t \widehat{\varepsilon}_t^2 \right) / (T-4)} \sim F(1, T-4),$$

or a *Likelihood Ratio test* ($LR - test$) computed as:

$$LR - test = 2 (MLL_{UR} - MLL_R) \sim \chi^2 (1),$$

where MLL_{UR} denotes the maximum log-likelihood of the unrestricted model (3), and MLL_R denotes the maximum log-likelihood of the restricted model (6).

- Suppose that a researcher estimates the static linear regression model and finds evidence of first order residual serial correlation. If the observed residual serial correlation is due to the omission of dynamics, then the appropriate model to use is the dynamic linear regression (1). If on the other hand, the observed residual serial correlation is due to first order autocorrelated disturbances, then the appropriate way to model y as a function of x is:

$$y_t = a + bx_t + u_t,\ u_t = \rho u_{t-1} + v_t,\ v_t \sim IN\left(0, \sigma^2\right). \tag{7}$$

The above is the static model with an error term (u_t) which has serial correlation of order one. After some algebraic manipulation[4] we can rewrite model (7) as:

$$y_t = a\left(1-\rho\right) + \rho y_{t-1} + bx_t - \rho b x_{t-1} + v_t. \tag{7'}$$

Now compare models (1) and (7'). What is the relationship between them? Observe that in both equations the same dependent variable (y_t) is regressed on the same set of variables (i.e. a constant, y_{t-1}, x_t, and x_{t-1}). However, in regression (1) we estimate four parameters $(\alpha_0, \alpha_1, \beta_0, \beta_1)$, whereas in regression (7') we only need to estimate three parameters (a, b, ρ). Therefore, (7') is a restricted version of (1); we need to impose one[5] restriction on model (1) to obtain model (7'). This particular restriction is called the **common factor restriction:**

$$\begin{aligned} H_0 &: \beta_1 = -\alpha_1 \beta_0; \\ H_0 &: \beta_1 \neq -\alpha_1 \beta_0. \end{aligned}$$

Since the above restriction is nonlinear we cannot use an $F - test$; we can only use an asymptotic testing procedure, like an $LR - test$, to test the

[4]

$$\begin{aligned} y_t &= a + bx_t + u_t \Rightarrow y_{t-1} = a + bx_{t-1} + u_{t-1} \Rightarrow \\ \rho y_{t-1} &= \rho a + \rho b x_{t-1} + \rho u_{t-1}; \\ y_t - \rho y_{t-1} &= a + bx_t + u_t - \rho a - \rho b x_{t-1} - \rho u_{t-1} \\ & \quad a\left(1-\rho\right) + bx_t - \rho b x_{t-1} + u_t - \rho u_{t-1} \Rightarrow \\ y_t &= a\left(1-\rho\right) + \rho y_{t-1} + bx_t - \rho b x_{t-1} + v_t, \end{aligned}$$

since $u_t = \rho u_{t-1} + v_t$.

[5] Note that the number of restrictions is equal to the number of parameters in the unrestricted model minus the number of parameters in the restricted one ($1 = 4 - 3$ in this case).

validity of the common factor restriction

$$LR - test = 2\left(MLL_{UR} - MLL_R\right) \sim \chi^2\left(1\right),$$

where MLL_{UR} denotes the maximum log-likelihood of the unrestricted model (1), and MLL_R denotes the maximum log-likelihood of the restricted model (7). Note that while (1) is estimated with least squares, (7) has to be estimated with generalised least squares (e.g. the Cochrane-Orcutt method) in order to take into account the serial correlation. (Try to estimate models (1) and (7) in MFIT using the dataset IE98.FIT. Which model do you prefer?) If we accept the restriction, then model (7) is preferred; otherwise the dynamic linear regression model (1) is the appropriate one to use.

2. Use of Dummy Variables

The simplest way to accommodate changes in the structure of a regression model is to use a *dummy variable*, i.e. a variable which takes the value one or zero.

The classic example is the case of the aggregate consumption function, in which rationing, saving campaigns, and so on, make wartime consumption behaviour different from peacetime behaviour. In what follows, C_t denotes the (natural) log of real consumption expenditure, and Y_t denotes the (natural) log of real personal disposable income. Without loss of generality, we will use the static linear regression model to demonstrate the use of a dummy variable. We shall distinguish between four different cases:

- The case in which peacetime and wartime consumption behaviour are assumed to be identical in all respects:

  $$C_t = \alpha + \beta Y_t + \varepsilon_t,\ t = 1, 2, ..., S, S+1, S+2, ..., T,$$

  where $1, 2, ..., S$ are the wartime years and $S+1, S+2, ..., T$ are the peacetime years.

- The case where we assume that the intercept of the consumption function changes during wartime but the slope parameter (i.e. the elasticity of consumption with respect to income, since the variables are in logs) stays the same:

  $$C_t = \alpha + b_0 D_t + \beta Y_t + \varepsilon_t,$$

where $D_t = \left\{ \begin{array}{l} 1,\ t = 1, 2, ..., S,\ \text{i.e. if it is wartime} \\ 0,\ t = S+1, S+2, ..., T,\ \text{(peacetime)} \end{array} \right\}$. A test of whether such a change is statistically significant is provided by a test of the null hypothesis $H_0 : b_0 = 0$. (One way to test is by employing a $t - test$.)

- The case in which the intercept is assumed to remain constant but the slope coefficient is different during wartime:

$$C_t = \alpha + \beta Y_t + b_1 (D_t Y_t) + \varepsilon_t.$$

  Again we can test the statistical significance of this structural change by using a $t - test$; the null hypothesis is $H_0 : b_1 = 0$.

- The case where both the intercept and the slope parameter are assumed to change during wartime:

$$C_t = \alpha + b_0 D_t + \beta Y_t + b_1 (D_t Y_t) + \varepsilon_t.$$

  A test of whether such a change is statistically significant is provided by a test of the null hypothesis $H_0 : b_0 = 0, b_1 = 0$.

Note that in all the above cases the variance of the error term is assumed to be the same in war and peace years. If we wish to test whether this is indeed the case, then we need to estimate two separate regressions and use the Goldfeld-Quandt test (see Lecture Notes 6).