Business School, Brunel University
MSc. EC5501/5509 Modelling Financial Decisions and Markets/Introduction to Quantitative Methods
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Lecture Notes 4

1. Linear Models and Matrix Algebra

Matrices are a compact and convenient way of writing down systems of linear equations. By using matrix algebra, the fundamental results in econometrics can be presented in an elegant and compact format.

In general, a system of m linear equations in n variables $(x_1, x_2, ..., x_n)$ can be arranged into the following format:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= d_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= d_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= d_m. \end{aligned} \tag{1}$$

The double-subscribted parameter symbol a_{ij} represents the coefficient appearing in the ith equation and attached to the jth variable.

There are three types of ingredients in the above equation system (1). The first is the set of coefficients a_{ij}; the second is the set of variables $x_1, x_2, ..., x_n$; and the last is the set of constant terms $d_1, d_2, ..., d_3$. Let us arrange the three sets as three rectangular arrays and label them, respectively, as A, x, and d :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}. \tag{2}$$

1.1. Some Terminology

Each of the three arrays in (2) constitutes a matrix. A **matrix** is defined as a rectangular array of numbers, parameters, or variables. The members of this array are referred to as the elements of the matrix. As a shorthand device, the array in matrix A can be written more simply as

$$A = [a_{ij}]\,, \quad \left\{ \begin{array}{c} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{array} \right\}.$$

Note that every matrix is an ordered set. As a simple example, given the linear equation system

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

we can write

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}.$$

The number of rows and the number of columns in a matrix together define the **dimension** of the matrix. Since matrix A contains m rows and n columns, it is said to be of dimension $m \times n$ (read: "m by n"). Note that the row number always precedes the column number.

With the matrices defined in (2), we can express the equation system (1) simply as

$$\underset{(m\times n)(n\times 1)}{Ax} = \underset{(m\times 1)}{d} \tag{3}$$

When the number of columns is equal to the number of rows $(m = n)$ then we have a **square matrix.** There are several particular types of square matrices that occur frequently in econometrics.

A **triangular matrix** is one that has only zeros either above or below the main diagonal. If the zeroes are above the diagonal, the matrix is lower triangular; If the zeroes are below the diagonal, the matrix is upper triangular. For example,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 10 & 0 \\ 16 & 8 & -4 \end{bmatrix}.$$

A **symmetric matrix** is one in which $a_{ij} = a_{ji}$ for all i and j. For example,

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 3 & 5 & 2 \\ 7 & 2 & 1 \end{bmatrix}.$$

A **diagonal matrix** is a square matrix whose only nonzero elements appear on the main diagonal, moving from upper left to lower right. For example,

$$A = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 4.5 & 0 \\ 0 & 0 & -6 \end{bmatrix}.$$

An **identity matrix,** $\underset{(n\times n)}{I}$ is a diagonal matrix with ones on the diagonal. For example,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.2. Vectors

Some matrices may contain only one column, such as x and d in (2). Such matrices are given the special name **column vectors.** The dimension of x is $(n \times 1)$. If we arranged the variables x_j in a horizontal array, though, there would result a $(1 \times n)$ matrix, which is called a **row vector.** For notation purposes, a row vector is often distinguished from a column vector by the use of a primed symbol:

$$\underset{(1\times n)}{x'} = \begin{bmatrix} x_1 & x_2 & ... & x_n \end{bmatrix}.$$

Observe that a vector (whether row or column) is an ordered n-tuple, and as such it may be interpreted as a point in an n-dimensional space.

To say whether two vectors (x and y) are equal, or in order to make any operations on them, they must have the same dimension $(n \times 1)$. They are then said to conform. In this case we have that

$$\begin{aligned}
x &= y \Leftrightarrow x_j = y_j, \text{ where } j = 1, 2, ..., n; && \text{(4a)} \\
z &= x \pm y \Leftrightarrow z_j = x_j \pm y_j, \text{ where } j = 1, 2, ..., n; && \text{(4b)} \\
w &= \lambda x \Leftrightarrow w_j = \lambda x_j, \text{ where } \lambda \text{ is a scalar, } j = 1, 2, ..., n; && \text{(4c)} \\
\underset{(1\times n)(n\times 1)}{x'y} &= \sum_{j=1}^{n} x_j y_j = x_1 y_1 + x_2 y_2 + ... + x_n y_n, \text{ or} && \text{(4d)} \\
\underset{(1\times n)(n\times 1)}{y'x} &= \sum_{j=1}^{n} x_j y_j = x_1 y_1 + x_2 y_2 + ... + x_n y_n. && \text{(4d')}
\end{aligned}$$

Equations (4d) and (4d') represent the **inner product** of two vectors and it . In other words, the inner product is the sum of products of corresponding elements, and hence the inner product is a scalar. If, for example, we prepare after a shopping trip a vector of quantities (q) purchased of n goods and a vector of their prices (p), listed in the corresponding order, then their inner product

$$\underset{(1\times n)(n\times 1)}{q'p} = \underset{(1\times n)(n\times 1)}{p'q} = \sum_{j=1}^{n} q_j p_j = q_1 p_1 + q_2 p_2 + ... + q_n p_n$$

will give the total purchase cost.

Multiplication of vectors: an $(m \times 1)$ column vector x, and a $(1 \times n)$ row vector y', yield a product matrix xy' of dimension $(m \times n)$. For example:

$$\begin{aligned}
\text{if } x &= \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \text{ and } y' = \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} \text{ then} \\
\underset{(2\times 1)(1\times 3)}{xy'} &= \begin{bmatrix} 3\,(1) & 3\,(4) & 3\,(5) \\ 2\,(1) & 2\,(4) & 2\,(5) \end{bmatrix} = \underset{(2\times 3)}{\begin{bmatrix} 3 & 12 & 15 \\ 2 & 8 & 10 \end{bmatrix}}.
\end{aligned}$$

The vector $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ represents a *null vector* (also called *zero vector*).

A set of vectors is **linearly dependent** if any one of them can be expressed as a linear combination of the remaining vectors; otherwise they are linearly independent. For example, the three vectors

$$x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \; y = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \text{ and } z = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

are linearly dependent because z is a linear combination of x and y :

$$3x - 2y = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = z.$$

Note that this last equation can be alternatively expressed as

$$3x - 2y - z = 0.$$

Therefore, we can say that k vectors $v_1, v_2, ..., v_k$ are linearly dependent if and only if there exists a set of scalars $\lambda_1, \lambda_2, ..., \lambda_k$ (not all zero) such that

$$\lambda_1 \underset{(n\times1)}{v_1} + \lambda_2 \underset{(n\times1)}{v_2} + ... + \lambda_k \underset{(n\times1)}{v_k} = \underset{(n\times1)}{0}.$$

1.3. Algebraic Manipulation of Matrices

- Matrices A and B are **equal** if and only if they have the same dimensions and each element of A equals the corresponding element of B :

$$A = B \text{ if and only if } a_{ij} = b_{ij} \text{ for all } i \text{ and } j.$$

- The **transpose** of a matrix A, denoted A', is obtained by creating the matrix whose jth row is the jth column of the original matrix (i.e. we interchange the rows and columns of the original matrix). For example,

$$A = \underset{(2\times3)}{\begin{bmatrix} 3 & 12 & 15 \\ 2 & 8 & 10 \end{bmatrix}}, \; A' = \underset{(3\times2)}{\begin{bmatrix} 3 & 2 \\ 12 & 8 \\ 15 & 10 \end{bmatrix}}.$$

  For any matrix A,

$$(A')' = A.$$

  The transpose of a column vector is a row vector, and vice-versa.

- The operation of **addition (subtraction)** is extended to matrices by defining

$$C = A \pm B = [a_{ij} \pm b_{ij}].$$

  Matrices cannot be added unless they have the same dimensions, in which case they are *conformable* for addition. A *zero matrix* or *null matrix* is one whose elements are all zero. In the addition of matrices, it plays the same role as the scalar 0 in scalar addition, i.e $A + 0 = A$. It follows that matrix addition is commutative

$$A + B = B + A,$$

associative

$$(A + B) + C = A + (B + C),$$

and that

$$(A + B)' = A' + B'.$$

- **Matrices are multiplied** using the *inner product.* For an $(m \times k)$ matrix A and a $(k \times n)$ matrix B, the product matrix

$$\underset{(m\times n)}{C} = \underset{(m\times k)(k\times n)}{AB}$$

is an $(m \times n)$ matrix whose ijth element is the inner product of row i of A and column j of B. In order to multiply two matrices, the number of columns in the first must be the same as the number of rows in the second, in which case, they are *conformable* for multiplication. For example,

$$\begin{aligned}
\underset{(2\times 3)(3\times 2)}{AB} &= \begin{bmatrix} 1 & 3 & 2 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 6 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1(2) + 3(1) + 2(0) & 1(4) + 3(6) + 2(5) \\ 4(2) + 5(1) + (-1)(0) & 4(4) + 5(6) + (-1)(5) \end{bmatrix} \\
&= \underset{(2\times 2)}{\begin{bmatrix} 5 & 32 \\ 13 & 41 \end{bmatrix}}.
\end{aligned}$$

Multiplication of matrices is generally not commutative. For example, in the preceding calculation, AB is a (2×2) matrix, while

$$\begin{aligned}
\underset{(3\times 2)(2\times 3)}{BA} &= \begin{bmatrix} 2(1) + 4(4) & 2(3) + 4(5) & 2(2) + 4(-1) \\ 1(1) + 6(4) & 1(3) + 6(5) & 1(2) + 6(-1) \\ 0(1) + 5(4) & 0(3) + 5(5) & 0(2) + 5(-1) \end{bmatrix} \\
&= \underset{(3\times 3)}{\begin{bmatrix} 18 & 26 & 0 \\ 25 & 33 & -4 \\ 20 & 25 & -5 \end{bmatrix}}.
\end{aligned}$$

In other cases, AB may exist, but BA may be undefined or, if it does exist, may have different dimensions, as in the preceding example. However, in general, even if AB and BA do have the same dimensions, they will not be equal. In view of this, we define **premultiplication** and **postmultiplication** of matrices. In the product AB, B is *premultiplied* by A, while A is *postmultiplied* by B. For any matrix or vector, A,

$$AI = A, \text{ and } IA = A,$$

though if A is not a square matrix, the two identity matrices are of different orders. A conformable matrix of zeros produces $A0 = 0$, although the dimensions of the result depend on the dimensions of the zero matrix. Some general rules for matrix multiplication are as follows:

$$\begin{array}{ll} \text{Associative law:} & (AB)\,C = A\,(BC)\,, \\ \text{Distributive law:} & A\,(B+C) = AB + AC, \\ \text{Transpose of a product:} & (AB)' = B'A'. \end{array}$$

Finally, **scalar multiplication** of a matrix is the operation of multiplying every element of the matrix by a given scalar. For example,

$$2 \underset{(3\times 2)}{\begin{bmatrix} 3 & 2 \\ 12 & 8 \\ 15 & 10 \end{bmatrix}} = \underset{(3\times 2)}{\begin{bmatrix} 6 & 4 \\ 24 & 16 \\ 30 & 20 \end{bmatrix}}.$$

- The following **matrix differentiation rules** are very useful in econometrics:

$$\begin{aligned} \frac{\partial\,(Ax)}{\partial x} &= A', \text{ where } A \text{ is } (m \times n) \text{ and } x \text{ is } (n \times 1)\,; \\ \frac{\partial\,(x'Ax)}{\partial x} &= 2Ax, \text{ where } A \text{ is a symmetric matrix.} \end{aligned}$$

1.4. Rank of a Matrix

If the maximum number of linearly independent rows that can be found in a matrix $(m \times n)$ is r, then the matrix is said to be of rank r (the rank also tells us the maximum number of linearly independent columns). The rank of a rectangular $(m \times n)$ matrix can at most be m or n, whichever is smaller. A square matrix $(n \times n)$ is said to have **full rank** when it has n linearly independent rows (columns). In this case we say that the $(n \times n)$ matrix is **nonsingular.** In other words, an $(n \times n)$ nonsingular matrix has n linearly independent rows (columns); it is of rank n. For example, the matrix

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{bmatrix}$$

is a singular matrix (it does not have full rank) because its rows (columns) are linearly dependent:

$$\begin{aligned} 2\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} - 1\begin{bmatrix} 6 & 8 & 10 \end{bmatrix} &= 0, \\ \text{or } -1\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} - 1\begin{bmatrix} 5 \\ 2 \\ 10 \end{bmatrix} &= 0. \end{aligned}$$

1.5. Solution of a System of Equations: Inverse Matrices and Determinants

For most of our applications, we shall consider only square systems of equations: $Ax = b$, where A is a square matrix. In what follows, therefore, we take m to equal n. Since the number of rows in A is the number of equations, while the number of columns in A is the number of variables, this is the familiar case of "n equations in n unknowns".

To solve the system $Ax = b$ for x, we need to find a square matrix, C, such that $CA = I$. If we premultiply the equation system by C we get

$$\underset{(n\times n)(n\times n)(n\times 1)}{CAx} = \underset{(n\times n)(n\times 1)}{Cb} \Rightarrow \underset{(n\times 1)}{x} = \underset{(n\times 1)}{Cb}.$$

If the matrix C exists, it is the **inverse** of A, denoted

$$C = A^{-1}.$$

From the definition,

$$A^{-1}A = I \Rightarrow AA^{-1} = I.$$

Note that for a given matrix A, the transpose A' always exists, whereas the inverse matrix A^{-1} may or may not exist.

- Squareness of a matrix is a necessary condition for the existence of its inverse.
- A matrix whose inverse exists is **nonsingular,** i.e. it has n linearly independent column (and row) vectors. If the matrix possesses no inverse it is called a singular matrix.
- If the inverse of a matrix exists, then it is unique.
- Properties of inverses:

$$\begin{aligned}\left(A^{-1}\right)^{-1} &= A, \\ (AB)^{-1} &= B^{-1}A^{-1}, \\ (A')^{-1} &= \left(A^{-1}\right)'.\end{aligned}$$

To ascertain whether a square matrix A is nonsingular we can make use of the concept of the **determinant**. It is denoted by $|A|$ and it is a uniquely defined scalar associated with matrix A. (Note that determinants are defined only for square matrices.)

- If the determinant is nonzero, then the matrix has full rank (it is nonsingular). If the determinant is zero, then the matrix is singular.

The determinant of a (2×2) matrix A is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

If $|A| \neq 0$, then the inverse of A is given by

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

1.6. Determinant and Inverse of an $(n \times n)$ Matrix (Optional)

Generally, an nth-order determinant can be evaluated using the *Laplace Expansion.* (See the attached pages at the end of the lecture notes, photocopied from A.C. Chiang, "Fundamental Methods of Mathematical Economics".)

If the determinant of an $(n \times n)$ matrix A is nonzero, then its inverse exists and is given by the ratio of the adjoint matrix of A (adj A) and its determinant:

$$A^{-1} = \frac{\text{adj } A}{|A|},$$

The adjoint of A is the transpose of the cofactor matrix C of A. The cofactor matrix C is obtained after replacing each element a_{ij} of A by the corresponding cofactor $|C_{ij}|$. So we have that:

$$\text{adj } A \equiv C' = \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots\cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots\cdots & |C_{n2}| \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ |C_{1n}| & |C_{2n}| & \cdots\cdots & |C_{nn}| \end{bmatrix}.$$

For example, consider the following matrix A :

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}.$$

Since $|A| = 99 \neq 0$, the inverse exists. the cofactor matrix is

$$C = \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}.$$

Therefore,

$$\text{adj } A \equiv C' = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix},$$

and the desired inverse matrix is

$$A^{-1} = A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{99} \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}.$$