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Inequality Constraints in the Fractionally Integrated GARCH Model

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ABSTRACT

In this article we derive necessary and sufficient conditions for the nonnegativity of the conditional variance in the fractionally integrated generalized autoregressive conditional heteroskedastic (p, d, q) (FIGARCH) model of the order $p \leq 2$ and sufficient conditions for the general model. These conditions can be seen as being analogous to those derived by Nelson and Cao (1992, *Journal of Business & Economic Statistics 10*, 229–235) for the GARCH(p, q) model. However, the inequality constraints which we derive for the FIGARCH model illustrate two remarkable properties of the FIGARCH model which are in contrast to the GARCH model: (i) even if all parameters are nonnegative, the conditional variance can become negative and (ii) even if all parameters are negative (apart from d), the conditional variance can be nonnegative almost surely. In particular, the conditions for the (1, d, 1) model substantially enlarge the sufficient parameter set provided by Bollerslev and Mikkelsen (1996, *Journal of Econometrics 73*, 151–184). The importance of the result is illustrated in an empirical application of the FIGARCH(1, d, 1) model to Japanese yen versus U.S. dollar exchange rate data.

KEYWORDS: inequality constraints, long-memory and fractionally integrated GARCH processes

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The empirical relevance of long memory conditional heteroscedasticity, which was initially addressed in the work of Ding et al. (1993) and Ding and Granger (1996), has emerged in a variety of studies of economic and financial time series. By now it is a widely accepted stylized fact that the empirical autocorrelation functions (ACFs) of the squared or absolute values of many macro and financial variables are characterized by a very slow decay, indicating long memory and persistence.

The linear autoregressive conditional heteroskedastic model (LARCH) by Robinson (1991) was the first model permitting long memory in the conditional variance. Subsequently, many researchers have proposed extensions of generalized ARCH (GARCH)-type models which can produce long-memory behavior. The fractionally integrated GARCH (FIGARCH) model by Baillie, Bollerslev, and Mikkelsen (1996) can definitely be considered as the most established model among those which have proved to be suitable to handle the typical data features in many empirical applications [see, e.g., Bollerslev and Mikkelsen (1996), Beine and Laurent (2003), Conrad and Karanasos (2005a,b)]. Alternative specifications were suggested by Davidson (2004), Giraitis, Robinson, and Surgailis (2004), Karanasos, Psaradakis, and Sola (2004), and Zaffaroni (2004). Recent research has been aimed at a better understanding of the properties of these well-established models; for instance, Karanasos, Psaradakis, and Sola (2004) derive convenient representations for the ACF of the squared values of long-memory GARCH (LMGARCH) processes, while in a related study Conrad and Karanasos (2006) derive expressions for the impulse response function (IRF) of the LMGARCH model. For an up-to-date overview of theoretical findings on long-memory GARCH processes, see Giraitis, Leipus, and Surgailis (2005). Finally, Baillie (1996) and Henry and Zaffaroni (2003) provide excellent surveys of major econometric work on long-memory processes and their applications in economics and finance.

As in the Bollerslev (1986) GARCH model, conditions on the parameters of the FIGARCH model have to be imposed to ensure the nonnegativity of the conditional variance. Originally, Bollerslev (1986) imposed conditions on the parameters of the GARCH(p, q) model, which were sufficient to ensure the nonnegativity of the conditional variance, but these conditions simply required the nonnegativity of all parameters in the conditional variance specification.

Nelson and Cao (1992) showed that the restrictions imposed by Bollerslev (1986) can be substantially relaxed. By investigating the ARCH(∞) representation of the process, they derive necessary and sufficient conditions for the GARCH(p, q) model with $p = 1$ or 2 and sufficient conditions for the general model. In particular, some of the parameters are allowed to have a negative sign. This is important since empirical findings [see Nelson and Cao (1992) and the references therein] suggest that for many financial time series typically the parameter associated with the second lag of the squared innovation in the GARCH specification has a negative sign. The Bollerslev (1986) conditions rule out this case and thereby unnecessarily limit the flexibility of the model. This is nicely illustrated by He and Teräsvirta (1999) who have shown that for the

GARCH(p, q) model with $\max\{p, q\} = 2$, these weaker conditions imply richer shapes of the ACF of the squared residuals.

An easy way to guarantee the nonnegativity of the conditional variance in the GARCH(p, q) model with $p \leq 2$ is, therefore, firstly to estimate the unrestricted model and then to validate the Nelson and Cao (1992) conditions only in case that there are parameter estimates with a negative sign. By now the Nelson and Cao (1992) conditions are implemented in econometric packages such as the financial analysis package for GAUSS, PcGive, S-Plus, Rats, and G@RCH.

To validate whether a set of parameters suffices for the nonnegativity of the conditional variance in the FIGARCH(p, d, q) model is substantially more difficult. In contrast to the GARCH model, it is possible that (i) the conditional variance becomes negative, although all the parameters are positive and (ii) the conditional variance is nonnegative almost surely (a.s.) for all t, although all the parameters are negative (apart from d). These two observations imply that—independent of the sign of the estimated parameters—the nonnegativity conditions should always be verified.

Bollerslev and Mikkelsen (1996) provide sufficient conditions for the FIGARCH(1, d, 1) model. These conditions are validated in programs such as the G@RCH package for Ox developed by Laurent and Peters (2002). Since these conditions are only sufficient, there exist parameter values for which the conditions are violated, but still the conditional variance will be nonnegative almost surely. No conditions (not even sufficient) are available for higher-order models.

> Of course, for the FIGARCH(p, d, q) model to be well-defined and the conditional variance positive almost surely for all t, all the coefficients in the infinite ARCH representation must be non-negative [Bollerslev and Mikkelsen (1996, p. 159)].

In this article we derive necessary and sufficient conditions for the FIGARCH(p, d, q) model of orders up to $p = 2$ and sufficient conditions for the general (p, d, q) model, which reduce an infinite number of inequalities to a finite number. Once the parameters are estimated, one can easily validate these conditions. The results for the $(1, d, 1)$ specification which is used most often in empirical applications are discussed in detail. We illustrate graphically how the necessary and sufficient conditions dramatically enlarge the feasible parameter set compared with the set given by the sufficient conditions provided by Bollerslev and Mikkelsen (1996). For models of higher order ($p \geq 3$), we derive sufficient constraints which require only mild conditions on the parameters of the process. However, in practical applications, one will rarely have to make use of a specification with $p > 2$. We provide an efficient algorithm for computing the coefficients in the ARCH(∞) representation which can be used if the sufficient conditions are violated. Plotting the sequence of coefficients indicates whether the conditional variance can become negative or not.

Availability of these inequality constraints is of importance to any researcher estimating FIGARCH models and in particular when utilizing parameter estimates to obtain volatility forecasts which are then employed, for example, for long-term option pricing or value-at-risk computations.

An empirical example illustrates the importance of our results. We estimate a FIGARCH(1, d, 1) model for Japanese yen versus U.S. dollar exchange rate data using the G@RCH package for Ox. The parameter estimates clearly fail to satisfy the Bollerslev and Mikkelsen (1996) conditions, which would lead any researcher relying on these conditions to reject the model. The set of parameters does, however, satisfy the necessary and sufficient conditions derived in this article and hence guarantees the nonnegativity of the conditional variance.

We should mention that the conditions derived in this article also apply to the LMGARCH model since the coefficients in the ARCH(∞) representations of the FIGARCH and the LMGARCH models coincide. Moreover, the results directly extend to the multivariate constant correlation FIGARCH model and to the fractionally integrated autoregressive conditional duration (FIACD) model proposed by Jasiak (1998) which requires the nonnegativity of the conditional duration time.

The article is organized as follows. Section 1 sets out the model of interest, assumptions, and notation. In Section 2, we derive the necessary and sufficient conditions for the nonnegativity of the conditional variance in the FIGARCH(p, d, q) process. Section 3 discusses the empirical example. In Section 4, we suggest future developments. All proofs are deferred to the appendix.

1 THE FRACTIONALLY INTEGRATED GARCH MODEL

Following Robinson (1991) and Zaffaroni (2004), we define an ARCH(∞) process $\{\varepsilon_t, t \in \mathbb{Z}\}$ by the equations

$$\varepsilon_t = Z_t \sqrt{h_t}, \tag{1}$$

where $\{Z_t, t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed random variables with $\mathbb{E}(Z_t) = 0$, $\sigma_Z^2 = \mathbb{E}(Z_t^2) < \infty$, and

$$h_t = \tilde{\omega} + \sum_{i=1}^{\infty} \psi_i \varepsilon_{t-i}^2. \tag{2}$$

The parameter $\sigma_Z^2 \in \mathbb{R}^+$ was introduced by Zaffaroni (2004) and relaxes the assumption that $\mathbb{E}(Z_t^2) = 1$ which is common in the GARCH literature. A major issue in specifying a valid ARCH(∞) process is to guarantee the nonnegativity of the conditional variance a.s. for all t. For this to hold, it must be assumed that $\tilde{\omega} \geq 0$ and $\psi_i \geq 0$ for all $i \geq 1$.[1] Now, define $v_t = \varepsilon_t^2 - \sigma_Z^2 h_t$ which is, by

[1] Requiring that $\psi_i \geq 0$ for all i implies that $\mathbb{P}(h_t < \tilde{\omega}) = 0$. Hence, $\tilde{\omega}$ is the lower bound for the conditional variance. Note that the same statement holds for the GARCH(p,q) model [see Nelson and Cao (1992)].

construction, a martingale-difference sequence with respect to the filtration generated by $\{\varepsilon_s, s \leq t\}$. Let $\Psi(L) = \sum_{i=1}^{\infty} \psi_i L^i$ with L being the lag operator, then ε_t^2 can be represented as

$$[1 - \sigma_Z^2 \Psi(L)]\varepsilon_t^2 = \sigma_Z^2 \tilde{\omega} + v_t. \tag{3}$$

From Equation (1), we have $\mathbb{E}[\varepsilon_t] = 0$, $\mathrm{cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$ for $j \geq 1$, and by Equation (3), $\mathbb{E}[\varepsilon_t^2] = (\sigma_Z^2 \tilde{\omega})/(1 - \sigma_Z^2 \Psi(1))$. Therefore, it follows that the covariance stationarity of ε_t in the ARCH(∞) model requires $\tilde{\omega}, \sigma_Z^2 \in \mathbb{R}^+$, and $\sigma_Z^2 \Psi(1) < 1$.

In the following, we explain how the Baillie, Bollerslev, and Mikkelsen (1996) FIGARCH and the Karanasos, Psaradakis, and Sola (2004) LMGARCH models relate to the ARCH(∞) model given by Equations (1) and (2) under specific assumptions on $\tilde{\omega}$, σ_Z^2, and for certain finite parameterizations of $\Psi(L)$.

Baillie, Bollerslev, and Mikkelsen (1996) introduce the FIGARCH*(p, d, q)* model by assuming $\sigma_Z^2 = 1$ and defining ε_t^2 via the well-known "ARMA in squares" representation[2]

$$(1 - L)^d \Phi(L)\varepsilon_t^2 = \omega + B(L)v_t, \tag{4}$$

for some $\omega \in \mathbb{R}^+$, $0 \leq d \leq 1$ and lag polynomials $\Phi(L)$ and $B(L)$ defined as

$$\Phi(L) = 1 - \sum_{i=1}^{q} \phi_i L^i \quad B(L) = 1 - \sum_{i=1}^{p} \beta_i L^i.$$

The FIGARCH model can be interpreted as a special case of Equation (2) with

$$\tilde{\omega} = \frac{\omega}{B(1)} \quad \text{and} \quad \Psi(L) = 1 - \frac{(1 - L)^d \Phi(L)}{B(L)}.$$

For any $0 < d < 1$, the ψ_i coefficients will be characterized by a slow hyperbolic decay implying persistent impulse response weights [see Conrad and Karanasos

[2] Baillie and Mikkelsen (1996) alternatively proposed the Fractionally Integrated Exponential GARCH (FIEGARCH) model which specifies the logarithm of the conditional variance as a fractionally integrated process. This formulation allows to model the so-called leverage effect and nests the Nelson (1991) Exponential GARCH (EGARCH) model as a special case when $d = 0$. Moreover, the conditional variance of the FIEGARCH is positive by construction, and hence no constraints on the parameters are required. A discussion of the moment and memory properties of the FIEGARCH model can be found in Giraitis, Leipus, and Surgailis (2005, p. 18). Despite the nice properties of the FIEGARCH model, it is evident that the FIGARCH model is much more popular in empirical applications. One reason might be that the leverage effect is primarily a short-run phenomenon. Therefore FIGARCH and FIEGARCH perform very similar in modeling the long-run features of e.g. stock market volatility. However, the FIEGARCH often encounters convergence problems in the estimation procedure due to the fact that the current conditional variance is a highly nonlinear function of lagged conditional variances. Moreover, to our knowledge, no distribution theory for the maximum likelihood estimator has been established even for the EGARCH with $d = 0$.

(2006)]. However, the Baillie, Bollerslev, and Mikkelsen (1996) specification with $0 < d < 1$ and $\sigma_Z^2 = 1$ is not compatible with the covariance stationarity of the ε_t, since in this case we have $\Psi(1) = 1$ and the above covariance stationarity condition is violated. For $0 < d < 1$, it is possible to obtain the covariance stationarity of the ε_t by assuming $\sigma_Z^2 < 1$, but as shown by Zaffaroni (2004), Theorem 2 and Remark 2.1, this implies absolute summability of the ACF of the ε_t^2, ruling out long memory in ε_t^2.

The FIGARCH model nests the Bollerslev (1986) GARCH model for $d = 0$. Then the condition $\sigma_Z^2\Psi(1) < 1$ reduces to the well-known covariance stationarity condition for ε_t stated in Bollerslev (1986): $\sum_{j=1}^{q} \phi_j < 1$. This specification implies exponentially decaying coefficients ψ_i which lead to an absolutely summable exponentially decaying ACF of ε_t^2 and hence to a short-memory process. On the other hand, the IGARCH model is obtained under the restriction $d = 1$. Then $\Psi(1) = 1$, and the model is again not covariance stationary.

A model which is closely related to the FIGARCH model was suggested by Karanasos, Psaradakis, and Sola (2004). They define the LMGARCH(p, d, q) model also by assuming $\sigma_Z^2 = 1$ but model the squared residuals in terms of deviations from $\omega \in \mathbb{R}^+$, that is, by the equation

$$(1 - L)^d \Phi(L)(\varepsilon_t^2 - \omega) = B(L)v_t. \tag{5}$$

This small modification makes the LMGARCH model being analogously defined to the Autoregressive Fractionally Integrated Moving Average (ARFIMA) model for the mean and has important implications for the properties of ε_t. Equations (1) and (5) imply $\mathbb{E}[\varepsilon_t] = 0$, $\mathrm{cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$ for $j \geq 1$, and $\mathbb{E}[\varepsilon_t^2] = \omega < \infty$. This means that the LMGARCH model specifies a covariance stationary ε_t process, although $\sigma_Z^2 = 1$ and $\Psi(1) = 1$ for any $0 < d < 1$. Moreover, the LMGARCH specification implies that the autocorrelations $\{\rho_m(\varepsilon_t^2), m = 1,2,\ldots\}$ satisfy $\rho_m(\varepsilon_t^2) = O(m^{2d-1})$. Hence, provided that the fourth moment of the ε_t is finite, ε_t^2 exhibits long memory for all $0 < d < 0.5$, in the sense that the series $\sum_{m=0}^{\infty} |\rho_m(\varepsilon_t^2)|$ is properly divergent [see Karanasos, Psaradakis and Sola (2004)]. In summary, the advantage of the LMGARCH model compared with the FIGARCH model is that it combines the covariance stationarity of the ε_t with the long memory in the ε_t^2. The question whether the LMGARCH and the FIGARCH models are strictly stationary or not is still open at present [see Giraitis, Leipus, and Surgailis (2005, p. 11)]. The LMGARCH model leads to an ARCH(∞) representation with $\tilde{\omega} = 0$ and $\Psi(L) = 1 - (1 - L)^d \Phi(L)/B(L)$.

Hence, both models obey ARCH(∞) coefficients generated from the expansion of

$$\Psi(L) = 1 - \frac{(1 - L)^d \Phi(L)}{B(L)} = \sum_{i=1}^{\infty} \psi_i L^i.$$

In the following section, we derive conditions on the parameters $(\beta_1, \ldots, \beta_p, d, \phi_1, \ldots, \phi_q)$ which guarantee that $\psi_i \geq 0$ for all $i \geq 1$. Since the ARCH(∞) coefficients are the same for the FIGARCH and the LMGARCH models, our results hold for both models. Moreover, even if $\sigma_Z^2 \neq 1$, this will not affect the coefficients in the ARCH(∞) representation, and hence our results hold for an even broader class of ARCH(∞) models than the FIGARCH and LMGARCH models. Because of the predominant role played by the FIGARCH model in the literature on empirical applications, we state all the results in the following section in terms of this model.

Before we present our results, we state further assumptions and introduce some more notation which we utilize in the proofs of all theorems. We assume that the inverse roots λ_i, $i = 1, \ldots, p$, of the polynomial $B(L)$ are real and $0 \neq |\lambda_i| < 1$ for $i = 1, \ldots, p$.[3] Additionally, we assume that the roots of $\Phi(L)$ lie outside the unit circle, and $\Phi(L)$ and $B(L)$ have no common roots.[4] The assumptions on the roots of $\Phi(L)$ and $B(L)$ imply that $\Phi(1) > 0$ and $B(1) > 0$.

The fractional differencing operator $(1 - L)^d$ is most conveniently expressed in terms of the hypergeometric function $H(\cdot)$

$$(1 - L)^d = H(-d,1;1;L) = \sum_{j=0}^{\infty} g_j L^j,$$

where the coefficients g_j are given by

$$g_j = f_j \cdot g_{j-1} = \prod_{i=1}^{j} f_i \qquad \text{with} \qquad f_j = \frac{j-1-d}{j} \quad \text{for } j = 1,2,\ldots$$

and $g_0 = 1$. Note that $f_1 = -d < 0$, $f_2 = (1-d)/2 > 0$, and $f_j > 0$ for all $j > 2$ and hence $g_j < 0$ for all $j \geq 1$. It is easy to see that $f_j < f_{j+1}$ and $f_j \to 1$ as $j \to \infty$.

Furthermore, for $i > q \geq 0$, we define $F_i = -\sum_{l=0}^{q} \phi_l \prod_{j=l}^{q-1} f_{i-j}$ with $\phi_0 = -1$ and $\prod_{j=0}^{-1} = 1$, $F_i < F_{i+1}$ and $F_i \to 1 - \phi_1 - \cdots - \phi_q > 0$ as $i \to \infty$.

Let $(\lambda_{(1)}, \lambda_{(2)}, \ldots, \lambda_{(p)})$ be an ordering of the roots λ_i and define $\Lambda_r = \sum_{i=1}^{r} \lambda_{(i)}$, $r \leq p$. Hence, it follows that

$$F_i^{(r)} = \Lambda_r F_{i-1} + F_i f_{i-q} \to (\Lambda_r + 1)(1 - \phi_1 - \cdots - \phi_q), \tag{6}$$

and the limit is positive, provided that $\Lambda_r > -1$.

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[3] Our analysis does not cover complex roots in $B(L)$. Since we could not find any article in which a FIGARCH model was estimated with complex roots, we expect the empirical relevance of this case to be rather small. However, most of the recursions we derive also hold for complex roots and hence in principle it is possible to extend our results in this direction.

[4] If $\Phi(L)$ and $B(L)$ have common roots, the FIGARCH process reduces to a model of lower order.

2 INEQUALITY CONSTRAINTS FOR FIGARCH(p, d, q) MODEL

In this section we derive the inequality constraints which are necessary and sufficient for the nonnegativity of the conditional variance in the FIGARCH(p, d, q) model with $p \leq 2$ and sufficient conditions for the general model. The inequality constraints provided in Bollerslev and Mikkelsen (1996) for the FIGARCH(1, d, 1) model are substantially relaxed. As a special case ($d = 0$), the results of Nelson and Cao (1992) can be obtained.

As mentioned before, the nonnegativity of the conditional variance requires that all ψ_i coefficients in the ARCH(∞) representation are nonnegative. In general, this would mean imposing infinitely many inequality constraints on the ψ_i. By investigating the sequence for the different models, we find that the infinite number of restrictions reduces to a finite number. This means that it suffices to check the nonnegativity of $\psi_1, \ldots, \psi_k$ to guarantee the nonnegativity of the conditional variance. To relate the ψ_i sequence to the parameters of the process, we have to find convenient representations of the coefficients as functions of the parameters.

2.1 FIGARCH(1, d, q)

We begin with deriving the inequality constraints for the FIGARCH(1, d, q) process. Then we discuss the empirically important examples of the FIGARCH (1, d, 1), FIGARCH(0, d, 1), and FIGARCH(1, d, 0) models in detail.

Theorem 1 *The conditional variance of the FIGARCH(1, d, q) is nonnegative a.s. iff*

Case 1: $0 < \beta_1 < 1$

1. $\psi_1, \ldots, \psi_{q-1} \geq 0$ *and*
2. *either* $\psi_q \geq 0$ *and* $F_{q+1} \geq 0$ *or for* $k > q + 1$ *with* $F_{k-1} < 0 \leq F_k$, *it holds that* $\psi_{k-1} \geq 0$.

Case 2: $-1 < \beta_1 < 0$

1. $\psi_1, \ldots, \psi_{q-1} \geq 0$ *and*
2. *either* $\psi_q \geq 0$, $\psi_{q+1} \geq 0$, *and* $F^{(1)}_{q+2} \geq 0$ *or for* $k > q + 2$ *with* $F^{(1)}_{k-1} < 0 \leq F^{(1)}_k$, *it holds that* $\psi_{k-1} \geq 0$ *and* $\psi_{k-2} \geq 0$.

In the proof of Theorem 1, we obtain an easily computable recursion for the ψ_i coefficients which can be used in practice to validate the requirements of the theorem for a given set of parameter estimates. It is clear that in the FIGARCH(1, d, q) model, it suffices to check $q + 1$ conditions if $\beta_1 > 0$ and $q + 2$ conditions if $\beta_1 < 0$ to ensure the nonnegativity of the conditional variance for all t.

Because the FIGARCH(1, d, 1) model is definitely the most often used specification in empirical applications, we intensively discuss the derivation of the corresponding inequalities and their interpretation. The ARCH(∞) representation of the FIGARCH(1, d, 1) model leads to following recursions (see proof of Theorem 1) for the corresponding ψ_i coefficients:

$$\psi_1 = d + \phi_1 - \beta_1 \tag{7}$$

$$\psi_i = \beta_1\psi_{i-1} + (f_i - \phi_1)(-g_{i-1}) \quad \text{for all } i \geq 2 \tag{8}$$

and alternatively,

$$\psi_i = \beta_1^2\psi_{i-2} + [\beta_1(f_{i-1}\phi_1) + (f_i - \phi_1)f_{i-1}](-g_{i-2}) \quad \text{for all } i \geq 3 \tag{9}$$

Corollary 1 *The conditional variance of the FIGARCH(1, d, 1) is nonnegative a.s. iff*

Case 1: $0 < \beta_1 < 1$

either $\psi_1 \geq 0$ *and* $\phi_1 \leq f_2$ *or for* $k > 2$ *with* $f_{k-1} < \phi_1 \leq f_k$, *it holds that* $\psi_{k-1} \geq 0$.

Case 2: $-1 < \beta_1 < 0$

either $\psi_1 \geq 0, \psi_2 \geq 0$, *and* $\phi_1 \leq f_2(\beta_1 + f_3)/(\beta_1 + f_2)$ *or for* $k > 3$ *with* $f_{k-2}(\beta_1 + f_{k-1})/(\beta_1 + f_{k-2}) < \phi_1 \leq f_{k-1}(\beta_1 + f_k)/(\beta_1 + f_{k-1})$, *it holds that* $\psi_{k-1} \geq 0$ *and* $\psi_{k-2} \geq 0$.

This corollary can be derived from the recursions by the following considerations. First, note that $-g_i > 0$ for $i \geq 1$. The proof uses then the fact that $F_i = f_i - \phi_1$ and $F_i^{(1)} = \beta_1(f_{i-1} - \phi_1) + (f_i - \phi_1)f_{i-1}$ are increasing and that for both expressions there exists a k such that $F_{k-1} < 0 \leq F_k$ and $F_{k-1}^{(1)} < 0 \leq F_k^{(1)}$. For example, consider Case 1. If $\psi_1 \geq 0$ and $\phi_1 \leq f_2$, this implies $\phi_1 < f_i$ for all $i > 2$ and hence the nonnegativity of all ψ_i by Equation (8). If $\phi_1 > f_2$, then there exists a k such that $\phi_1 \leq f_k$, and so ψ_{k-1} implies $\psi_i \geq 0$ for all $i \geq k$ because f_i is increasing. Also, $\psi_{k-1} \geq 0$ and $f_{k-1} < 0$ imply $\psi_i \geq 0$ for all $i \leq k-2$. Case 2 can be treated analogously using Equation (9).

Next, we compare Corollary 1 with the already existing sufficient conditions for the FIGARCH(1, d, 1) model suggested in Baillie, Bollerslev, and Mikkelsen (1996), Bollerslev and Mikkelsen (1996), and Chung (1999). Baillie, Bollerslev and Mikkelsen (1996, p. 22) provide the following sufficient constraints

$$0 \leq \beta_1 \leq \phi_1 + d \qquad \text{and} \qquad 0 \leq d \leq 1 - 2\phi_1$$

which are equivalent to $\psi_1 \geq 0$ and $F_2 \geq 0$. Alternatively, Bollerslev and Mikkelsen (1996, p. 159) state the inequality constraints

$$\beta_1 - d \leq \phi_1 \leq \frac{2-d}{3} \qquad \text{and} \qquad d\left[\phi_1 - \frac{1-d}{2}\right] \leq \beta_1(\phi_1 - \beta_1 + d)$$

which are equivalent to ψ_1, $\psi_2 \geq 0$ and $F_3 \geq 0$. Hence, these inequality constraints reflect the first condition in Case 1 of Corollary 1 or the arbitrary choice of $k = 3$ (again Case 1). The Bollerslev and Mikkelsen (1996) conditions are weaker than the Baillie, Bollerslev, and Mikkelsen (1996) conditions but restrict $\phi_1 \leq f_3$. Note that both sets of sufficient conditions do not cover Case 2 where $-1 < \beta_1 < 0$, since the corollary requires $F_3^{(1)} \geq 0$. Finally, Chung (1999) suggests a third set of sufficient constraints which is given by

$$0 \leq \phi_1 \leq \beta_1 \leq d < 1 \tag{10}$$

and provides two examples:

(i) $\phi_1 = 0.6$, $\beta_1 = 0.7$, and $d = 0.8$

(ii) $\phi_1 = 0.5$, $\beta_1 = 0.2$, and $d = 0.25$.

The first set of parameters satisfies Equation (10) but not the Bollerslev and Mikkelsen (1996) conditions, while the second satisfies the Bollerslev and Mikkelsen (1996) conditions but not Equation (10). Chung (1999, p. 18) concludes "The examples show that there may be parameter values that cannot satisfy either set of sufficient conditions while still allow all ψ_i coefficients to be positive."

The corollary above provides necessary and sufficient conditions and thereby solves this problem. One can easily check that the parameters in both examples satisfy the conditions of Corollary 1. Moreover, in comparison with the Bollerslev and Mikkelsen (1996) sufficient conditions, it widens the range of admissible parameters: (i) if $f_{k-1} < \phi_1 \leq f_k$ with $k > 3$ parameters can still be admissible and (ii) we allow for $\beta_1 < 0$.

Figure 1 illustrates how the inequality constraints from Corollary 1, Case 1, extend the sufficient set from Bollerslev and Mikkelsen (1996) to the necessary and sufficient set for two fixed values of d, that is, for $d \in \{0.1, 0.9\}$ and $\phi_1 > 0$.[5] The set denoted B + M is given by the Bollerslev and Mikkelsen (1996) conditions, while the set denoted C + H is the area which is allowed for by Corollary 1, Case 1, but not by the Bollerslev and Mikkelsen (1996) conditions. The dashed line separating the two sets corresponds to $\phi_1 = f_3$. For a given value of d, f_3 is the upper bound for ϕ_1 in the Bollerslev and Mikkelsen (1996) conditions. The

[5] Note that we exclude $\phi_1 = \beta_1$ by the assumption that $\Phi(L)$ and $B(L)$ have no common roots.

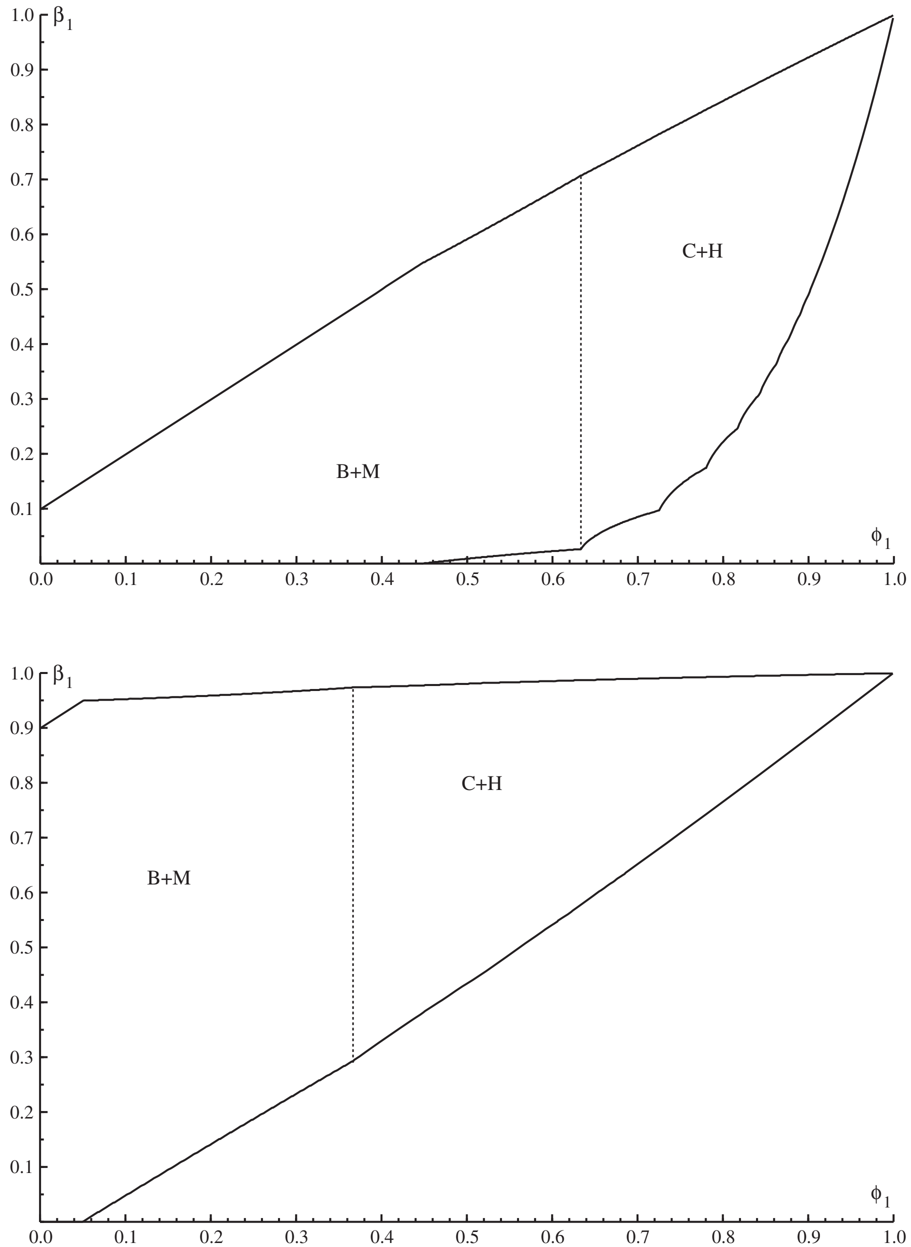

Figure 1 Necessary and sufficient parameter set for FIGARCH(1, d, 1) model (Case 1 and $\phi_1 > 0$) with $d = 0.1$ (upper) and $d = 0.9$ (lower).

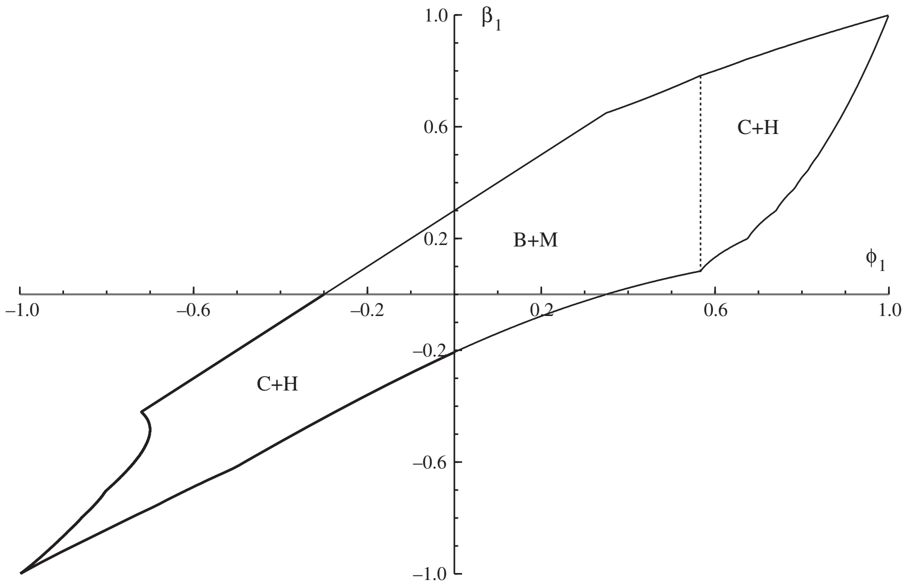

Figure 2 Necessary and sufficient parameter set for FIGARCH(1, d, 1) model with $d = 0.3$.

joint set, that is, $B + M \cup C + H$, is the necessary and sufficient one. The necessary and sufficient set given by Figure 2 covers Case 1 and Case 2 for $d = 0.3$ and $-1 < \phi_1 < 1$. As can be easily seen, Corollary 1 dramatically enlarges the set of parameter values which is allowed for and thereby allows for a greater flexibility in model specification. In particular, note that in contrast to the GARCH model, the conditional variance of the FIGARCH can be nonnegative although $\phi_1 < 0$ and $\beta_1 < 0$, and on the other hand, it can become negative although all parameters are positive. When d is approaching 0, the parameter set described by Corollary 1 converges to the well-known necessary and sufficient set for a GARCH(1, 1) model with parameters $\alpha_1 = \phi_1 - \beta_1$ and β_1. When d approaches 1, the FIGARCH(1, d, 1) collapses to a GARCH(1, 2) model with parameters $\alpha_1 = 1 + \phi_1 - \beta_1$, $\alpha_2 = -\phi_1$ and β_1, which add to 1. The admissible parameter set again coincides with the parameter set given by Nelson and Cao (1992) for this model.

For the GARCH(2, 2) model, He and Teräsvirta (1999) show that the Nelson and Cao (1992) necessary and sufficient conditions imply richer shapes of the ACF of the squared residuals compared with shapes implied by the Bollerslev's (1986) sufficient conditions. They discover four possible types of ACFs which can be generated. Type 1 is characterized by a smooth monotonic decay from ρ_1 onward, while type 2 reaches its peak at $\rho_2 > \rho_1$ with monotonic decay from ρ_2 onward. The autocorrelations may be oscillating with peak at either ρ_1 or ρ_2, which are the Cases 3 and 4. Expressions for the

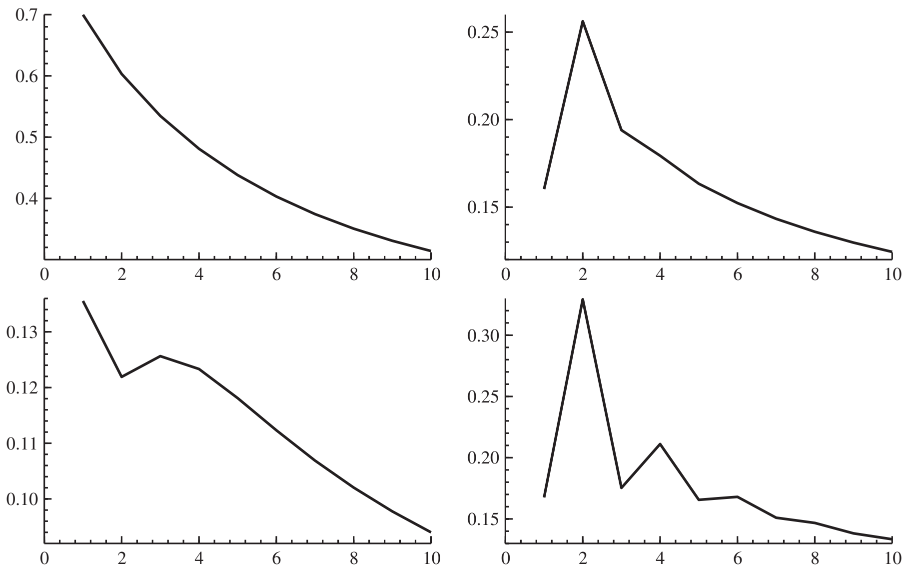

Figure 3 Four types of autocorrelation functions for LMGARCH(1, d, 1) model with $d = 0.3$. Type 1: $\phi_1 = 0.7$, $\beta_1 = 0.5$ (upper left), type 2: $\phi_1 = -0.2$, $\beta_1 = 0.05$ (upper right), type 3: $\phi_1 = 0.3$, $\beta_1 = 0.53$ (lower left), and type 4: $\phi_1 = -0.5$, $\beta_1 = -0.25$ (lower right).

ACF of the squared residuals in the LMGARCH(p, d, q) were derived in Karanasos, Psaradakis, and Sola (2004).[6] While the long-run behavior of the ACF is governed by the fractional differencing parameter d, the short-run behavior is determined by ϕ_1 and β_1. We plot in Figure 3 the ACFs of a LMGARCH(1, d, 1) model with $d = 0.3$ and certain combinations of parameters ϕ_1 and β_1 lying in the necessary and sufficient set (see Figure 2). It is evident that even the LMGARCH(1, d, 1) model can generate all four types of ACFs described by He and Teräsvirta (1999) for the GARCH(2, 2) model. Interestingly, type 4 is generated by parameter values which do not lie in the Bollerslev and Mikkelsen (1996) sufficient set, and we could not find any combination of parameters with $\beta_1 > 0$ leading to this type. This finding suggests that the enlarged parameter set directly translates into an increased flexibility in characterizing the autocorrelation structure of the squared residuals.

Example 1 Baillie, Han, and Kwon (2002) can serve as an example which illustrates the importance of our result. ARFIMA-FIGARCH(1, d, 1) models are estimated to several inflation series. In Table 3, p. 507, the estimated parameters for the French inflation data are $\widehat{\beta}_1 = 0.899$, $\widehat{d} = 0.331$, and $\widehat{\phi}_1 = 0.859$. Even though these parameters do not satisfy the Bollerslev and Mikkelsen (1996) conditions

[6] Recall that in contrast to the FIGARCH, the LMGARCH is covariance stationary. However, as pointed out by Karanasos, Psaradakis, and Sola (2004), both models have the same "second-order structure."

(clearly $\widehat{\phi}_1 > \widehat{f}_3$), the parameters are in accordance with the conditions given by Corollary 1, Case 1.

Remark 1 *Corollary 1 can be directly applied to the bivariate constant correlation FIGARCH(1, d, 1) model given by the equations*

$$h_{11,t} = \frac{\omega_{11}}{1-\beta_{11}} + \left[1 - \frac{(1-\phi_{11}L)(1-L)^{d_1}}{(1-\beta_{11}L)}\right]\varepsilon^2_{1,t},$$

$$h_{22,t} = \frac{\omega_{22}}{1-\beta_{22}} + \left[1 - \frac{(1-\phi_{22}L)(1-L)^{d_2}}{(1-\beta_{22}L)}\right]\varepsilon^2_{2,t},$$

and $h_{12,t} = \rho\sqrt{h_{11,t}h_{22,t}}$.

Positive definiteness of the variance–covariance matrix is guaranteed if and only if $|\rho| < 1$ *and the parameters* $(\phi_{jj}, d_j, \beta_{jj})$ *satisfy the condition given in Corollary 1 for* $j = 1, 2$. *This model is used, for example, by Brunetti and Gilbert (2000) to investigate long memory in oil price data. Brunetti and Gilbert (2000) check the sufficient Bollerslev and Mikkelsen (1996) constraints for each equation.*

Remark 2 *Jasiak (1998) extends the Engle and Russel (1998) ACD(p, q) model to the FIACD(p, d, q) model, that is, he assumes that the duration time* x_i *between the i-th and* $(i-1)$*-th event can be modeled as*

$$x_i = \delta_i Z_i \quad \text{and} \quad \delta_i = \frac{\omega}{1-\beta_1} + \left[1 - \frac{(1-\phi_1 L)(1-L)^d}{(1-\beta_1 L)}\right]x_i^2$$

with Z_i *being a sequence of i.i.d random variables with expectation one and* δ_i *the conditional expectation of the i-th duration. Applied in this context, Corollary 1 ensures the nonnegativity of the conditional duration time* δ_i.

The FIGARCH(1, d, 1) nests two interesting submodels: the FIGARCH(1, d, 0) and the FIGARCH(0, d, 1). Although Corollary 1 does not explicitly cover the cases with $\phi_1 = 0$ and $\beta_1 = 0$, they can be treated along the same lines of argumentation.

Corollary 2 *The conditional variance of the FIGARCH(0, d, 1) model is nonnegative a.s. iff*

1. $\psi_1 \geq 0 \Leftrightarrow d + \phi_1 \geq 0$
2. $F_2 \geq 0 \Leftrightarrow (1-d)/2 - \phi_1 \geq 0$

If $\beta_1 = 0$, the recursion given by Equation (8) reduces to $\psi_i = (f_i - \phi_1)(-g_{i-1})$ for $i \geq 2$. We impose $\psi_1 \geq 0$. Recall that $-g_i > 0$ for $i \geq 1$. The nonnegativity of ψ_2 requires $f_2 - \phi_1 \geq 0$. Since f_i is increasing, $\psi_2 \geq 0$ implies $\psi_i \geq 0$ for all $i > 2$.[7]

Example 2 Conrad and Karanasos (2005b) estimate ARFIMA–FIGARCH models for the inflation rates of ten European countries. For Belgium, the preferred specification for the conditional variance is a FIGARCH(0, d, 1) model with estimated parameters $\widehat{d} = 0.330$ and $\widehat{\phi}_1 = -0.280$. Since $\widehat{d} > -\widehat{\phi}_1$ and $\widehat{\phi}_1 < 0$, it follows from Corollary 2 that these parameters guarantee the nonnegativity of the conditional variance for all t. The autocorrelation structure implied by these parameters is of type 2.

Corollary 3 *The conditional variance of the FIGARCH(1, d, 0) model is nonnegative a.s. iff*[8]

Case 1: $0 < \beta_1 < 1$

$$\psi_1 \geq 0 \Leftrightarrow d - \beta_1 \geq 0$$

Case 2: $-1 < \beta_1 < 0$

$$\psi_2 \geq 0 \Leftrightarrow \left(d - \sqrt{2(2-d)}\right)/2 \leq \beta_1$$

If $\phi_1 = 0$, the recursions given by Equations (8) and (9) reduce to $\psi_i = \beta_1\psi_{i-1} - g_i$ for $i \geq 2$ and $\psi_i = \beta_1^2\psi_{i-2} + (\beta_1 + f_i)(-g_{i-1})$ for $i \geq 3$. For $0 < \beta_1 < 1$, assuming $\psi_1 \geq 0$ together with $-g_i \geq 0$ for all $i \geq 1$ implies $\psi_i \geq 0$ for all i. For $-1 < \beta_1 < 0$, it is easy to see that $\psi_2 \geq 0$ implies $F_2^{(1)} = \beta_1 + f_2 \geq 0$. Since f_i is increasing, it follows that $F_i^{(1)} = \beta_1 + f_i > 0$ for all $i \geq 3$. Hence, $\psi_1 \geq 0$ (ensured by $\beta_1 < 0$) and $\psi_2 \geq 0$ imply $\psi_i \geq 0$ for all i.

Above we illustrated the consequences of our less severe parameter constraints on the shapes of the ACF for the LMGARCH(1, d, 1) model. The implications of allowing for $\beta_1 < 0$ on the degree of persistence that can be modeled are now illustrated by considering the ACF and the IRF of the LMGARCH(1, d, 0) model. As an example, we assume $d = 0.45$. According to Corollary 3, the range of values for β_1 which guarantee the nonnegativity of the conditional variance is given by $-0.1925 \leq \beta_1 \leq 0.45$. The LMGARCH(1, d, 0) model with the restriction $\beta_1 \geq 0$ cannot produce ACFs in the area above the ACF with $\beta_1 = 0$, as can be seen from Figure 4 (upper), that is, the degree of

[7] Similarly, from the proof of Theorem 1, it follows that the conditional variance of the FIGARCH(0, d, q) model is nonnegative a.s. iff *1.* $\psi_1, \ldots, \psi_q \geq 0$ and *2.* $F_{q+1} \geq 0$. Note that $F_{q+1} \geq 0$ is trivially fulfilled, if $\phi_i \leq 0$ for $i = 1, \ldots, q$.

[8] Note that in Baillie, Bollerslev, and Mikkelsen (1996, p. 11), only Case 1 was considered, and $0 \leq \beta_1 < d \leq 1$ is stated as a necessary and sufficient condition for the conditional variance of the FIGARCH(1, d, 0) model to be positive a.s. for all t.

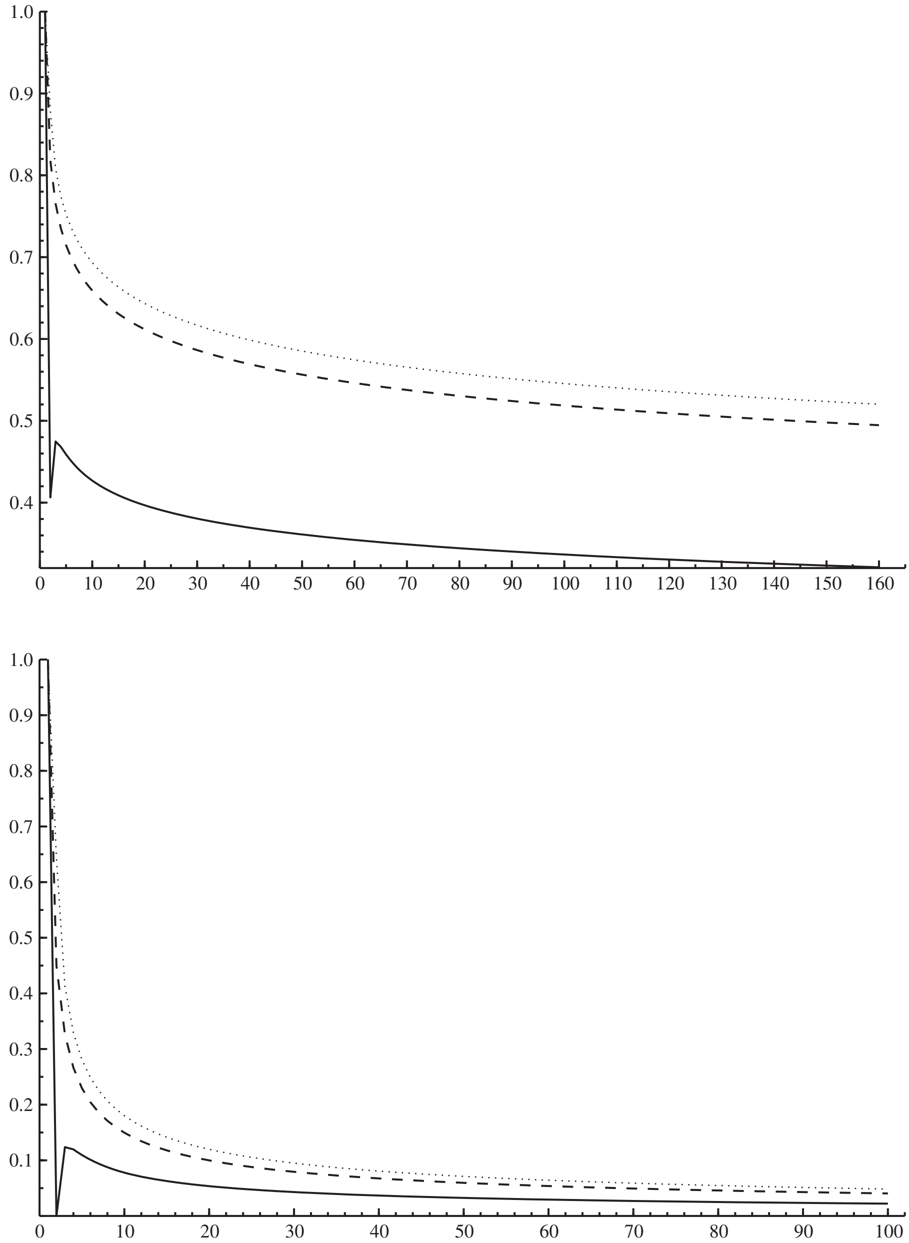

Figure 4 ACF of ε_t^2 (upper) and impulse response function (IRF) function (lower) for LMGARCH (1, d, 0) with $d = 0.45$ and $\beta_1 = 0.45$ (solid), $\beta_1 = 0$ (dashed) and $\beta_1 = -0.1925$ (dotted), respectively.

persistence is unnecessarily limited. Expressions for the IRF of the LMGARCH(p, d, q) model have been obtained by Conrad and Karanasos (2006). Figure 4 (lower) plots the IRF of the LMGARCH(1, d, 0) with $d = 0.45$ and $\beta \in \{-0.1925, 0, 0.45\}$. Clearly, allowing for $\beta_1 < 0$ increases the flexibility of the IRF. In accordance with the result for the ACF, a negative β_1 increases the persistence of the process.

2.2 FIGARCH(2, *d*, *q*)

Before we consider cases with $p \geq 2$, we derive a recursive representation of the $\{\psi_i\}$ sequence. Again, let $(\lambda_{(1)}, \ldots, \lambda_{(p)})$ be some ordering of the λ_i. We make use of the representation in the proofs of the subsequent theorems.

Lemma 1 *The sequence $\{\psi_i, i = 1, 2, \ldots\}$ can be written as*

$$\psi_i = \psi_i^{(p)} \qquad \textit{where}$$
$$\psi_i^{(r)} = \lambda_{(r)}\psi_{i-1}^{(r)} + \psi_i^{(r-1)} \qquad 1 < r \leq p,\ i \geq 1,$$

and the sequence of $\{\psi_i^{(1)}\}$ is given by

$$\psi_i^{(1)} = -c_i + \sum_{j=1}^{\min\{i,q\}} \phi_j c_{i-j} \qquad \textit{for } i = 1, \ldots, q \quad \textit{and with} \quad c_i = \sum_{j=0}^{i} \lambda_{(1)}^{i-j} g_j$$
$$\psi_i^{(1)} = \lambda_{(1)}\psi_{i-1}^{(1)} + F_i(-g_{i-q}) \qquad \textit{for } i > q$$

with starting values $\psi_0^{(r)} = -1$, $r = 1, \ldots, p$.

Now, we turn to the case $p = 2$. Without loss of generality, we assume that $\lambda_1 \geq \lambda_2$. No inequality constraints–not even sufficient—have been established for $p \geq 2$ in the literature on long-memory GARCH models so far. We first consider the FIGARCH(2, d, 0) model and then combine the results with those from the FIGARCH(1, d, q) model.

Proposition 1 *The conditional variance of the FIGARCH(2, d, 0) model is nonnegative a.s. iff (recall that in this case $\lambda_1 + \lambda_2 = \beta_1$ and $\lambda_1 \cdot \lambda_2 = -\beta_2$)*

Case 1: $1 > \lambda_1 \geq \lambda_2 > 0$, that is, $\beta_1 > 0$, $\beta_2 < 0$

$$\psi_1 \geq 0 \Leftrightarrow d \geq \lambda_1 + \lambda_2 = \beta_1$$

Case 2: $1 > \lambda_1 > 0 > \lambda_2 > -1$ *and* $\lambda_1 \geq |\lambda_2|$, *that is,* $\beta_1 \geq 0$, $\beta_2 > 0$

$$\psi_1 \geq 0 \textit{ and } \psi_2 \geq 0$$

Case 3: $1 > \lambda_1 > 0 > \lambda_2 > -1$ *and* $\lambda_1 < |\lambda_2|$, *that is,* $\beta_1 < 0$, $\beta_2 > 0$

either if $\psi_2^{(1)} \geq 0 \Leftrightarrow \lambda_2(d - \lambda_2) + f_2 d \geq 0$ *or* $\psi_2, \psi_4, \ldots, \psi_{\bar{k}-2} \geq 0$, *where* $\bar{k} = \min_{\tilde{k} \text{ even}} \{\psi_{\tilde{k}}^{(1)} > 0\}$ *with* $\lambda_{(1)} = \lambda_2$ *and* $\lambda_{(2)} = \lambda_1$.

Case 4: $0 > \lambda_1 \geq \lambda_2 > -1$, *that is,* $\beta_1 < 0$, $\beta_2 < 0$

$\psi_2^{(1)} \geq 0$ *and* $\psi_2 \geq 0$ *where* $\lambda_{(1)} = \lambda_1$ *and* $\lambda_{(2)} = \lambda_2$.

Proposition 1, Case 1, states that the conditional variance can be nonnegative, although $\beta_2 < 0$. Case 3 shows that with $p = 2$, we can allow for $\beta_1 < 0$ [in the GARCH(2, 2) model, one needs at least $\beta_1 > 0$]. Finally, Case 4 illustrates that in contrast to the GARCH model with $p = 2$, where at least one root must be nonnegative, in the FIGARCH with $p = 2$ we can allow for both roots being negative. Note, however, that this case will rarely appear for financial data in practice since estimating $\beta_1 < 0$ and $\beta_2 < 0$ is very unlikely. We are not aware of any application where such a parameter combination has been estimated.

Example 3 Beine and Laurent (2003) estimate AR(1)-FIGARCH(2, *d*, 0) models for the exchange rate of the Japanese yen, French franc, and British pound against the U.S. dollar. The parameter estimates for all the three currencies presented in Table 2, p. 651, are such that Case 2 applies. Checking $\widehat{\psi}_1$ and $\widehat{\psi}_2$ immediately proves that all $\widehat{\psi}_{\mathrm{i}}$ coefficients are nonnegative and thereby confirms the validity of the chosen model specifications.

Theorem 2 *The conditional variance of the FIGARCH(2, d, q) model is nonnegative a.s. iff*

Case 1: $1 > \lambda_1 \geq \lambda_2 > 0$, *that is,* $\beta_1 > 0$, $\beta_2 < 0$

1. $\psi_1 \geq 0$, *and for* $q \geq 2$: $\psi_1, \ldots, \psi_{q-1} \geq 0$
2. *either* $\psi_q^{(1)} \geq 0$ *and* $F_{q+1} \geq 0$ *or for k with* $F_{k-1} < 0 \leq F_k$ *either* $\psi_{k-1}^{(1)} \geq 0$ or $\psi_{\bar{k}-1} \geq 0$, *where*

$$\bar{k} = \min_{\tilde{k} > k} \left\{ -\lambda_{(1)} \psi_{k-1}^{(1)} < \sum_{j=0}^{\tilde{k}} F_{k+j}(-g_{k-q+j}) \lambda_{(1)}^{-j} \right\}$$

and $\lambda_{(1)} = \lambda_1$, $\lambda_{(2)} = \lambda_2$.

Case 2: $1 > \lambda_1 > 0 > \lambda_2 > -1$, $\lambda_1 \geq |\lambda_2|$, *that is,* $\beta_1 \geq 0$, $\beta_2 > 0$

1. $\psi_1 \geq 0$, *and for* $q \geq 2$: $\psi_1, \ldots, \psi_{q-1} \geq 0$ *and*
2. *either* $\psi_q \geq 0$ *and* $F_{q+1} \geq 0$ *or for* $k \geq q + 2$ *with* $F_{k-1} < 0 \leq F_k$, *we have* $\psi_1, \ldots, \psi_{k-1} \geq 0$.

Case 3: $1 > \lambda_1 > 0 > \lambda_2 > -1$, $\lambda_1 < |\lambda_2|$, *i.e.* $\beta_1 < 0, \beta_2 > 0$

1. $\psi_1 \geq 0$, *and for* $q \geq 2$: $\psi_2, \ldots, \psi_{q-1} \geq 0$
2. *either* $\psi_q^{(1)} \geq 0$, $\psi_{q+1}^{(1)} \geq 0$, *and* $F_{q+2}^{(1)} \geq 0$ *or for* k *with* $F_{k-1}^{(1)} < 0 \leq F_k^{(1)}$ *either* $\psi_{k-1}^{(1)} \geq 0$ *and* $\psi_{k-2}^{(1)} \geq 0$ *or* $\psi_{q+1}, \ldots, \psi_{\bar{k}-1} \geq 0$, *where*

$$\bar{k} = \min_{\tilde{k}>k} \left\{ -\lambda_{(1)}^2 \psi_{k-2}^{(1)} < \sum_{j=0}^{\tilde{k}} F_{k+2j}^{(1)}(-g_{k-q+2j-1})\lambda_{(1)}^{-2j}, \right.$$
$$\left. -\lambda_{(1)}^2 \psi_{k-1}^{(1)} < \sum_{j=0}^{\tilde{k}} F_{k+2j}^{(1)}(-g_{k-q+2j-1})\lambda_{(1)}^{-2j} \right\}$$

and $\lambda_{(1)} = \lambda_2$, $\lambda_{(2)} = \lambda_1$.

Case 4: $0 > \lambda_1 \geq \lambda_2 > -1$, $\lambda_1 + \lambda_2 > -1$, *that is*, $-1 < \beta_1 < 0$, $\beta_2 < 0$

1. $(\lambda_1, d, \phi_1, \ldots, \phi_q)$ *satisfy the conditions for the FIGARCH(1, d, q) model*
2. $\psi_1, \ldots, \psi_{k-1} \geq 0$, *where* $k > q+2$ *is s.t.* $F_{k-1}^{(2)} < 0 \leq F_k^{(2)}$

Case 5: $0 > \lambda_1 \geq \lambda_2 > -1$, $\lambda_1 + \lambda_2 \leq -1$ *that is*, $\beta_1 \leq -1$, $\beta_2 < 0$

1. $(\lambda_{(1)}, d, \phi_1, \ldots, \phi_q)$ *satisfy the conditions for the FIGARCH(1, d, q) model*
2. *There exists a* $\bar{k}$ *such that* $S_{\bar{k},k} > 0$, *where* k *such that* $F_{k-1}^{(1)} < 0 \leq F_k^{(1)}$ *and*

$$S_{j,i} = \lambda_{(1)}\Lambda_2 \sum_{l=1}^{j} \lambda_{(1)}^{2(i-l)} F_{i+2l}^{(1)}(-g_{i-q+2l-1}) + F_{i+2j}^{(2)}(-g_{i-q+2j-1})$$

3. $\psi_1, \ldots, \psi_{\bar{k}} \geq 0$

Again, note that Cases 4 and 5 are not of empirical interest, and in particular Case 5 which implies $\beta_1 < -1$. Moreover, Theorem 2 illustrates that with increasing p the number of cases which have to be analyzed grows exponentially. As the conditions are already quite complex when $p = 2$, the analysis becomes even worse when $p \geq 3$.

Remark 3 [Relation to GARCH(p, q)] *Since the simple GARCH model is nested within the FIGARCH, we treat it as a special case. If we set* $d = 0$, *we obtain*

$$\Psi(L) = \frac{B(L) - \Phi(L)}{B(L)} = \frac{\alpha(L)}{B(L)} = \sum_{i=1}^{\infty} \psi_i L^i.$$

In analogy to Lemma 1, the $\{\psi_i\}$ *sequence can be obtained in the GARCH(p, q) as*

$$\psi_i = \psi_i^{(p)} \quad \text{where}$$
$$\psi_i^{(r)} = \lambda_{(r)}\psi_{i-1}^{(r)} + \psi_i^{(r-1)} \text{ for } 1 < r \le p,\ i \ge 1 \text{ with } \psi_0^{(r)} = 0,$$

where $\psi_i^{(1)}$ *in the GARCH(1, q) is given by* $\psi_i^{(1)} = \sum_{j=1}^{i} \lambda_{(1)}^{i-j}\alpha_j$ *for* $i = 1, 2, \ldots, q$, *and* $\psi_i^{(1)} = \lambda_{(1)}^{i-q}\psi_q$ *for all* $i > q$, *and for some ordering* $(\lambda_{(1)}, \ \ldots, \lambda_{(p)})$.

For $p = 1$ *or* $p = 2$, *it can be easily shown that our methodology leads to the same necessary and sufficient nonnegativity constraints as were derived by Nelson and Cao (1992).*

2.3 FIGARCH(*p*, *d*, *q*)

As in the GARCH(p, q) model, necessary and sufficient conditions are more difficult to derive for the general FIGARCH(p, d, q) process. Instead, we state two sets of sufficient conditions. A first set which is more restrictive on the parameters but—when satisfied—immediately implies $\psi_i \ge 0$ for all i. A second set which is less restrictive on the parameters but requires to check the nonnegativity of the first k ψ_i.

Theorem 3 *The conditional variance of the FIGARCH(p, d, q) model is nonnegative a.s. iff*

1. *Let s be the number of inverse roots of B(L) which are positive. If p is even we require* $s \ge p/2 - 1$ *and if p is odd we require* $s \ge (p-1)/2$.
2. $\psi_1 = d + \phi_1 - (\lambda_1 + \lambda_2 + \cdots + \lambda_p) \ge 0$.
3. a. *If* $p = s$, *then there must be a* λ_i *s.t.* $(\lambda_i, d, \phi_1, \ldots, \phi_q)$ *satisfy the conditions for the FIGARCH(1, d, q) model*

   b. *If* $p > s$, *then there must exist an ordering* $(\lambda_{(1)}, \lambda_{(2)}, \ldots, \lambda_{(p)})$ *of the roots* λ_i *s.t.*

   (i) $(\lambda_{(1)}, d, \phi_1, \ldots, \phi_q)$ *satisfy the conditions for the FIGARCH(1, d, q) model*

   (ii) $0 > \lambda_{(2(p-s-\mathbf{1})+1)} \ge \lambda_{(2(p-s-1-\mathbf{1})+1)} \ge \ldots \ge \lambda_{(5)} \ge \lambda_{(3)}$ *and* $\lambda_{(2)} \ge |\lambda_{(3)}|$, $\lambda_{(4)} \ge |\lambda_{(5)}|, \ldots, \lambda_{(2(p-s-\mathbf{1}))} \ge |\lambda_{(2(p-s-\mathbf{1})+1)}|$

   (iii) $\psi_2^{(3)}, \psi_2^{(5)}, \ldots, \psi_2^{(2(p-s-\mathbf{1})+1)} \ge 0$

   where $\mathbf{1} = 1$, *if* $\lambda_{(1)} < 0$ *and 0 otherwise.*

In a slightly modified version, the same arguments can be applied to the GARCH(p, q) model using the representation given in Remark 3. Such a sufficient condition is more restrictive than the sufficient condition stated in Nelson and Cao (1992), which is given by $\lambda_{(1)} > \max_{i=2,\ldots,p}\{|\lambda_{(i)}|\}$, but—in contrast to the Nelson and Cao (1992) condition—directly implies $\psi_i \geq 0$ for all i.

Now, we come to the second and less restrictive sufficient condition. For this condition, we have to find $0 \leq p_1 \leq p_2 \leq p$ with $p_2 - p_1$ even, such that the ordering of the p inverse roots of $B(L)$ is in the following way

$$\begin{aligned}
&\lambda_{(1)} \leq \cdots \leq \lambda_{(p_1)} < 0\\
&\lambda_{(p_1+1)} > 0,\ \lambda_{(p_1+2)} < 0, \ldots, \lambda_{(p_2-1)} > 0,\ \lambda_{(p_2)} < 0\\
&\qquad\qquad \text{with } \lambda_{(p_1+2i-1)} + \lambda_{(p_1+2i)} \geq 0, \quad i = 1, \ldots, (p_2 - p_1)/2\\
&\lambda_{(p_2+1)} \geq \cdots \geq \lambda_{(p)} > 0
\end{aligned}$$

This ordering is of course not unique as there can always be taken positive and negative roots to build a new pair as well as pairs can be separated, such that p_1 and p_2 differ. But it is always possible to find such an ordering.

Theorem 4 *If in the FIGARCH(p, d, q), there exists an ordering of the roots such that* $\Lambda_{p_1} > -1$ *then there exists a k such that* $\psi_i \geq 0$ for all $i > k$.

From this theorem, it is clear that if $\Lambda_{p_1} > -1$ it is sufficient to check a finite number of ψ_i, $i \leq k$ to find out if for specific parameter values the conditional variance is nonnegative a.s. for all t. The existence of such a k under a weak condition is a strong result, since ψ_i is an *i-th* order polynomial in all parameters. The unknown k can be found going along the proof of this theorem. This procedure can easily be implemented. However, the condition is not necessary, that is, it is possible to find parameter values such that $\psi_i \geq 0$ for all i and $\Lambda_{p_1} < -1$ for every ordering of the roots. The set which is not covered by this theorem is expected to be small, for example, in the FIGARCH(2, d, q), the theorem would cover four of the five cases considered in Theorem 2.

Since for economic data we always would expect that $\Lambda_p = \beta_1 > 0$, it suffices to check a finite number of coefficients. In this case, the theorem provides a necessary and sufficient condition.

For higher-order models—in which the estimated parameters do not satisfy the sufficient condition given by Theorem 4—the sequence ψ_i can be calculated using Lemma 1. By plotting the sequence for sufficiently high lags, one can obtain an indication whether the conditional variance will stay positive or not. However, this does of course not guarantee the nonnegativity of the conditional variance for all t.

Theorem 4 can be seen as being analogous to the sufficient condition stated in Nelson and Cao (1992) for the GARCH(p, q) model.

3 EMPIRICAL EXAMPLE

To illustrate the importance of our results, we investigate an empirical time series. We employ daily exchange rate data for the Japanese yen versus, U.S. dollar sourced from the Datastream database for the period November 1, 1993 to November 18, 2003, giving a total of 2621 observations. The continuously compounded returns are computed as $r_t = 100 \cdot [\log(p_t) - \log(p_{t-1})]$ where p_t is the price on day t. Table 1 presents the quasi-maximum likelihood parameter estimates for a FIGARCH(1, d, 1) model $(r_t = \mu + \varepsilon_t)$ estimated with the G@RCH package.

Table 1 FIGARCH(1, d, 1) estimates for Japanese yen versus U.S. dollar exchange rate

$\widehat{\mu}$	$\widehat{\omega}$	$\widehat{d}$	$\widehat{\phi}_1$	$\widehat{\beta}_1$
0.012 (0.928)	0.027 (0.896)	0.264 (2.899)	0.592 (2.131)	0.727 (2.821)

Note: Numbers in parentheses are t-statistics.

In addition to these parameter estimates, G@RCH provides the following output:

The positivity constraint for the FIGARCH (1, d, 1) is not observed [see Bollerslev and Mikkelsen (1996) for more details].

Obviously, $\widehat{\phi}_1 > \widehat{f}_3$ in this case, and thus the Bollerslev and Mikkelsen (1996) conditions are violated. We check the conditions from Corollary 1, since it allows for $\widehat{\phi}_1 > \widehat{f}_3$. Step 1 is to determine k which is given by $(1+\widehat{d})/(1-\widehat{\phi}_1) \leq k$, and so Step 2 is to verify that $\widehat{\psi}_{k-1} \geq 0$ which suffices for $\widehat{\psi}_i \geq 0$ for all $i = 1, 2, \ldots$. For our empirical example, we find $k = 4$ and so $\widehat{\psi}_3$ has to be calculated by the recursions for the (1, d, 1) model. It can be easily seen that $\widehat{\psi}_3 > 0$ for our parameter estimates. Hence, the set of parameters guarantee that the conditional variance is nonnegative a.s. for all t. Figure 5 illustrates the set of necessary and sufficient $(\phi_1 > 0, \beta_1 > 0)$ parameter values for $\widehat{d} = 0.264$. The dashed line bounds the Bollerslev and Mikkelsen (1996) set and is given by $\phi_1 = \widehat{f}_3$. The cross represents the estimated parameter combination which lies in the necessary and sufficient set.[9] The practitioner solely relying on the G@RCH output—and hence on the Bollerslev and Mikkelsen (1996) conditions—would have falsely rejected the model.

4 CONCLUSIONS

In this article, we derive necessary and sufficient conditions which ensure the nonnegativity of the conditional variance in the FIGARCH model of the order $p \leq 2$ and sufficient conditions for the general model. These conditions are

[9] Note that the estimated parameters also violate the sufficient constraints suggested by Chung (1999).

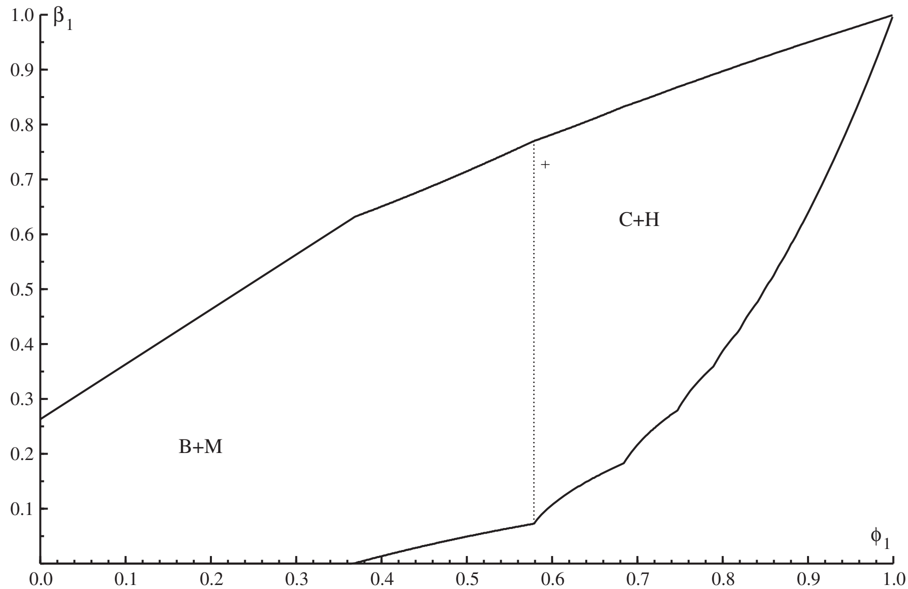

Figure 5 Necessary and sufficient parameter set for FIGARCH(1, d, 1) model (Case 1 and $\phi_1 > 0$) with $\widehat{d} = 0.264$.

important since any practitioner estimating FIGARCH models—in particular, when using parameter estimates for forecasting volatility—has to make sure that the model is well defined in the sense that it cannot lead to negative conditional variances. This issue is even more important when considering long-memory GARCH models since—unlike with the short-memory GARCH models—one cannot easily deduce the nonnegativity of the conditional variance from the sign of the estimated parameters. So far only a sufficient condition for the FIGARCH(1, d, 1) model was available, and no conditions existed for higher-order models. We demonstrate graphically how the necessary und sufficient conditions for the (1, d, 1) model enlarge the feasible parameter set which has important implications for the permitted shapes of the ACFs and IRFs of the LMGARCH model and thereby widens the range of data features that can be handled. The lack of knowledge concerning conditions which ensure the nonnegativity of the conditional variance in higher-order models is presumably one reason why these models have been applied rarely in the literature. Studies as Caporin (2003) which are concerned with identification and order selection in long-memory GARCH models restrict their analysis to the set of parameters defined by Bollerslev and Mikkelsen (1996) and hence to models of orders (1, d, 1), (1, d, 0), and (0, d, 0) only. Our work is intended to close this gap. As with the Nelson and Cao (1992) conditions, we suggest that econometric packages should state not only the estimated parameters but also whether those satisfy the necessary and sufficient conditions derived in this article. An interesting avenue for future research would be to analyze the implications of imposing the necessary and sufficient restrictions directly on the maximum likelihood estimation.

Our results extend to more sophisticated long-memory specifications such as the asymmetric power FIGARCH, the multivariate constant correlation FIGARCH model, and long-memory ACD models in which it must be ensured that the conditional duration time does not take negative values.

APPENDIX

We begin by deriving a recursive representation of the ψ_i sequence for the FIGARCH(1, d, q) model. This representation will be used to prove Theorem 1. In the FIGARCH(1, d, q) model, we can write

$$\begin{aligned}\Psi(L) &= 1 - \Phi(L)(1-L)^d(1-\beta_1 L)^{-1} \\ &= 1 - (1 - \phi_1 L^1 - \phi_2 L^2 - \cdots - \phi_q L^q) \cdot \sum_{i=0}^{\infty} c_i L^i \\ &= 1 - \sum_{i=0}^{\infty} \left(c_i - \phi_1 c_{i-1} - \cdots - \phi_q c_{i-q}\right) L^i = \sum_{i=1}^{\infty} \psi_i L^i\end{aligned}$$

where

$$c_i = \sum_{j=0}^{i} \beta_1^{i-j} g_j \quad \text{for} \quad i \geq 0 \quad \text{and} \quad c_i = 0 \quad \text{for} \quad i < 0.$$

Hence, the sequence $\{\psi_i,\ i = 1,2,\ldots\}$ can be written as

$$\psi_i = -c_i + \sum_{j=1}^{\min\{i,q\}} \phi_j c_{i-j} \quad \text{for } i > 0.$$

Note that the following recursion applies: $c_i = \beta_1 c_{i-1} + g_i$.

Proof of Theorem 1

Case 1: $0 < \beta_1 < 1$

"$\Leftarrow$"

1. $\psi_1, \ldots, \psi_{q-1} \geq 0$ by assumption.
2. (i) If $\psi_q \geq 0$ and $F_{q+1} \geq 0$, this ensures $\psi_i \geq 0$ for all $i > q$, since F_i is increasing and

$$\begin{aligned}\psi_i &= -c_i + \phi_1 c_{i-1} + \cdots + \phi_q c_{i-q} \quad \text{for } i \geq q+1 \\ &= -(\beta_1 c_{i-1} + g_i) + \phi_1(\beta_1 c_{i-2} + g_{i-1}) + \cdots + \phi_q(\beta_1 c_{i-q-1} + g_{i-q}) \\ &= \beta_1(-c_{i-1} + \phi_1 c_{i-2} + \cdots + \phi_q c_{i-q-1}) \\ &\quad + (f_i f_{i-1} \ldots f_{i-q+1} - \phi_1 f_{i-1} \ldots f_{i-q+1} - \cdots - \phi_q)(-g_{i-q}) \\ &= \beta_1 \psi_{i-1} + F_i(-g_{i-q}) \geq 0. \end{aligned} \tag{A.1}$$

(ii) If $F_{q+1} < 0$, then for ψ_i with $q < i < k$ it holds that

$$\psi_i = \beta_1 \psi_{i-1} + F_i(-g_{i-q}) \geq 0 \Leftrightarrow \beta_1 \psi_{i-1} \geq F_i g_{i-q} > 0 \Rightarrow \psi_{i-1} \geq 0.$$

Thus, $\psi_i \geq 0$ implies $\psi_{i-1} \geq 0$. As $\psi_{k-1} \geq 0$, it follows recursively that $\psi_i \geq 0$ for all $q \leq i < k$. For $i \geq k$, we have

$$\psi_i = \beta_1 \psi_{i-1} + F_i(-g_{i-q})$$

and hence, from $\psi_{k-1} \geq 0$ follows $\psi_k \geq 0$ since $F_k \geq 0$. $\Rightarrow$ $\psi_i \geq 0$ for all $i > k$ by induction.

"$\Rightarrow$"

1. The first condition and ψ_q, $\psi_{k-1} \geq 0$ are trivially fulfilled.
2. Either $F_{q+1} \geq 0$ or $F_{q+1} \leq 0$, but since $F_{i-1} \leq F_i$ and $F_i \to 1 - \sum_{j=1}^{q} \phi_i > 0$, there exists a k s.t. $F_i \geq 0$ for all $i \geq k$. □

Case 2: $-1 < \beta_1 < 0$

"$\Leftarrow$"

1. $\psi_1, \ldots, \psi_{q-1} \geq 0$ by assumption.
2. We make use of the following recursion
$$\psi_i = \beta_1^2 \psi_{i-2} + F_i^{(1)}(-g_{i-q-1}) \text{ for } i \geq q+2\,. \tag{A.2}$$
3. (i) If $F_{q+2}^{(1)} \geq 0$, then $\psi_q \geq 0$ and $\psi_{q+1} \geq 0$ ensure that $\psi_i \geq 0$ for all $i \geq q+2$.

   (ii) If $F_{q+2}^{(1)} < 0$, then for ψ_i with $q+2 < i < k$ it holds that

$$\begin{aligned} \psi_i &= \beta_1^2 \psi_{i-2} + F_i^{(1)}(-g_{i-q-1}) \geq 0 \\ \Leftrightarrow \beta_1^2 \psi_{i-2} &\geq -F_i^{(1)}(-g_{i-q-1}) \geq 0 \\ \Rightarrow \psi_{i-2} &\geq 0. \end{aligned}$$

   Thus, $\psi_i \geq 0$ implies $\psi_{i-2} \geq 0$. As $\psi_{k-1} \geq 0$ and $\psi_{k-2} \geq 0$, it follows recursively that $\psi_i \geq 0$ for all $q \leq i < k$. For $i \geq k$, we use Equation (A.2) and hence, from $\psi_{k-1} \geq 0$ and $\psi_{k-2} \geq 0$, it follows that $\psi_i \geq 0$ for all $i > k$ by induction.

"⇒"

1. The first condition and ψ_q, ψ_{q+1}, ψ_{k-2}, $\psi_{k-1} \geq 0$ are trivially fulfilled.
2. Either $F^{(1)}_{q+1} \geq 0$ or $F^{(1)}_{q+1} \leq 0$, but since $F^{(1)}_{i-1} \leq F^{(1)}_i$ and $F^{(1)}_i \to \beta_1(1-\phi_1-\cdots-\phi_q)+(1-\phi_1-\cdots-\phi_q) > 0$, there exists a k s.t. $F^{(1)}_i \geq 0$ for all $i \geq k$. □

Proof of Lemma 1

$$\begin{aligned}
\Psi(L) &= 1-\frac{\Phi(L)(1-L)^d}{B(L)} = 1-\Phi(L)(1-L)^d(1-\lambda_{(1)}L)^{-1}\cdot\cdots\cdot(1-\lambda_{(p)}L)^{-1} \\
&= 1+\sum_{i=0}^{\infty}\psi_i^{(1)}L^i\cdot\sum_{i=0}^{\infty}\lambda_{(2)}^i L^i\cdot\cdots\cdot\sum_{i=0}^{\infty}\lambda_{(p)}^i L^i \\
&= 1+\sum_{i=0}^{\infty}\psi_i^{(2)}L^i\cdot\sum_{i=0}^{\infty}\lambda_{(3)}^i L^i\cdot\cdots\cdot\sum_{i=0}^{\infty}\lambda_{(p)}^i L^i \\
&\vdots \\
&= 1+\sum_{i=0}^{\infty}\psi_i^{(p)}L^i \\
&= \sum_{i=1}^{\infty}\psi_i^{(p)}L^i
\end{aligned}$$

since

$$\sum_{i=0}^{\infty}\psi_i^{(r-1)}L^i\cdot\sum_{i=0}^{\infty}\lambda_{(r)}^i L^i = \sum_{i=0}^{\infty}\psi_i^{(r)}L^i$$

with

$$\begin{aligned}
\psi_i^{(r)} &= \sum_{j=0}^{i}\lambda_{(r)}^j\psi_{i-j}^{(r-1)} = \psi_i^{(r-1)}+\sum_{j=1}^{i}\lambda_{(r)}^j\psi_{i-j}^{(r-1)} = \psi_i^{(r-1)}+\lambda_{(r)}\sum_{j=0}^{i-1}\lambda_{(r)}^j\psi_{i-j-1}^{(r-1)} \\
&= \psi_i^{(r-1)}+\lambda_{(r)}\psi_{i-1}^{(r)}.
\end{aligned}$$

□

Using the representation from Lemma 1 and Equation (A.1) for $\psi_i^{(1)}$, we deduce for $i > q+1$

$$\psi_i^{(r)} = \sum_{k=0}^{r-1}\psi_{i-2}^{(r-k)}\lambda_{(r-k)}(\Lambda_r-\Lambda_{r-k-1})+F_i^{(r)}(-g_{i-q-1}). \tag{A.3}$$

Repeated application of Equation (A.3) leads to

$$
\psi_{i+2m}^{(r)} = \lambda_{(r)}^{2m}\psi_i^{(r)} + \sum_{k=1}^{r-1}\sum_{j=1}^{m}\lambda_{(r)}^{2(m-j)}\lambda_{(r-k)}(\Lambda_r - \Lambda_{r-k-1})\psi_{i+2j-2}^{(r-k)} + \sum_{j=1}^{m}\lambda_{(r)}^{2(m-j)}F_{i+2j}^{(r)}(-g_{i-q+2j-1}) \tag{A.4}
$$

for $m = 1, 2, \ldots$ and $i > q + 1$.

We will make use of Equations (A.3) and (A.4) in the subsequent proofs.

Proof of Proposition 1

"$\Leftarrow$"

Case 1: $1 > \lambda_1 \geq \lambda_2 > 0$.

Set $\lambda_{(1)} = \lambda_1$, $\lambda_{(2)} = \lambda_2$ and note that $\psi_i^{(1)}$ is identical with ψ_i from the FIGARCH(1, d, 0), Case 1, since $\lambda_{(1)} = \lambda_1 > 0$.

Observe that $\psi_1 = \psi_1^{(2)} = d - (\lambda_{(1)} + \lambda_{(2)}) \geq 0$ implies $\psi_1^{(1)} = d - \lambda_{(1)} \geq 0$. Furthermore, from Proposition 3, Case 1, we know that $\psi_1^{(1)} \geq 0$ implies $\psi_i^{(1)} \geq 0$ for all i. Hence, it follows that $\psi_i = \lambda_{(2)}\psi_{i-1} + \psi_i^{(1)} \geq 0$ for all $i \geq 2$.

Case 2: $1 > \lambda_1 > 0 > \lambda_2 > -1$ and $\lambda_1 \geq |\lambda_2|$.

Set $\lambda_{(1)} = \lambda_1$, $\lambda_{(2)} = \lambda_2$. Then for $i > 2$, we can write

$$
\begin{aligned}
\psi_i &= \lambda_{(2)}\psi_{i-1} + \psi_i^{(1)} \\
&= \lambda_{(2)}\psi_{i-1} + \lambda_{(1)}\psi_{i-1}^{(1)} - g_i \\
&= \lambda_{(2)}\psi_{i-1} + \lambda_{(1)}(\psi_{i-1} - \lambda_{(2)}\psi_{i-2}) - g_i \\
&= (\lambda_{(1)} + \lambda_{(2)})\psi_{i-1} - \lambda_{(1)}\lambda_{(2)}\psi_{i-2} - g_i.
\end{aligned}
$$

Since $\lambda_{(1)} + \lambda_{(2)} \geq 0$, $\lambda_{(1)}\lambda_{(2)} < 0$ and $g_i < 0$, it suffices to assume that $\psi_1 \geq 0$ and $\psi_2 \geq 0$ to ensure that $\psi_i \geq 0$ for all $i > 2$.

Case 3: $1 > \lambda_1 > 0 > \lambda_2 > -1$ and $\lambda_1 < |\lambda_2|$.

Note that $\psi_i^{(1)}$ is identical with ψ_i from the FIGARCH(1, d, 0) model, Case 2, since $\lambda_{(1)} = \lambda_2 < 0$.

1. $\psi_1^{(1)} = d - \lambda_{(1)} \geq 0$ is obviously satisfied.
2. (i) If $\lambda_{(1)}$ is such that $\psi_2^{(1)} \geq 0$ which implies $\psi_i^{(1)} \geq 0$ for all i (Proposition 3, Case 2), we immediately obtain $\psi_i \geq 0$ for all i.

   (ii) If $\lambda_{(1)}$ is such that $\psi_2^{(1)} \leq 0$, we make use of the recursion

$$
\psi_i^{(1)} = \lambda_{(1)}^2\psi_{i-2}^{(1)} + (\lambda_{(1)} + f_i)(-g_{i-1}) \text{for } i \geq 3. \tag{A.5}
$$

Since we know that there exists a k s.t. $\lambda_{(1)} + f_{k-1} < 0 \leq \lambda_{(1)} + f_k$, we can conclude that there exists an even $\bar{k} = k + 2i + \mathbf{1}_k$ with

$$\psi^{(1)}_{k+2i+\mathbf{1}_k} = \lambda^{2i+2}_{(1)}\psi^{(1)}_{k-2+\mathbf{1}_k} + \sum_{j=0}^{i}(\lambda_{(1)} + f_{k+2j+\mathbf{1}_k})(-g_{k-1+2j+\mathbf{1}_k})\lambda^{2(i-j)}_{(1)} \geq 0$$

since

$$0 \leq -\lambda^2_{(1)}\psi^{(1)}_{k-2+\mathbf{1}_k} < \sum_{j=0}^{i}(\lambda_{(1)} + f_{k+2j+\mathbf{1}_k})(-g_{k-1+2j+\mathbf{1}_k})\lambda^{-2j}_{(1)}$$

where the RHS is diverging, and $\mathbf{1}_k$ is defined as

$$\mathbf{1}_k = \begin{cases} 1 & \text{if } k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

By definition $\psi^{(1)}_4, \psi^{(1)}_6, \ldots, \psi^{(1)}_{\bar{k}-2} \leq 0$ and from $\psi^{(1)}_i = \lambda_{(1)}\psi^{(1)}_{i-1} - g_i$, it follows that $\psi^{(1)}_3, \psi^{(1)}_5, \ldots, \psi^{(1)}_{\bar{k}-1} \geq 0$.

Again from Equation (A.5), we deduce that $\psi^{(1)}_i \geq 0$ for all $i \geq \bar{k} + 1$. Observe that $\psi_1 = d - (\lambda_{(1)} + \lambda_{(2)}) \geq 0$ without further assumptions. For $i \geq 3$, we can apply the following recursion

$$\begin{aligned}\psi_i &= \lambda_{(2)}\psi_{i-1} + \psi^{(1)}_i \\ &= \lambda^2_{(2)}\psi_{i-2} + \lambda_{(2)}\psi^{(1)}_{i-1} + \psi^{(1)}_i \\ &= \lambda^2_{(2)}\psi_{i-2} + (\lambda_{(1)} + \lambda_{(2)})\psi^{(1)}_{i-1} - g_i.\end{aligned} \tag{A.6}$$

Given that $\psi_1 \geq 0$ and knowing that all $\psi^{(1)}_i \leq 0$ with i even, we obtain $\psi_i \geq 0$ for all $i < \bar{k}$ and i odd. By observing that $\psi^{(1)}_i \geq 0$ for all $i \geq \bar{k}$, we conclude from Equation (A.6) that $\psi_i \geq 0$ for all $i \geq \bar{k}$. It remains to assume that $\psi_2, \psi_4, \ldots, \psi_{\bar{k}-2} \geq 0$.

Case 4: $0 > \lambda_1 \geq \lambda_2 > -1$.

Note that $\psi^{(1)}_i$ is identical with ψ_i from the FIGARCH(1, d, 0) model, Case 2 with $\lambda_{(1)} = \lambda_1$. Again, observe that $\psi_1 = d - (\lambda_{(1)} + \lambda_{(2)}) \geq 0$ without further assumptions. Now consider ψ_2:

$$\begin{aligned}\psi_2 &= \lambda_{(2)}\psi_1 + \psi^{(1)}_2 \\ &= \lambda_{(2)}(d - (\lambda_{(1)} + \lambda_{(2)})) + \lambda_{(1)}(d - \lambda_{(1)}) + f_2 d \\ &= (\lambda_{(1)} + \lambda_{(2)} + f_2)d - \lambda^2_{(1)} - \lambda_{(2)}(\lambda_{(1)} + \lambda_{(2)})\end{aligned}$$

$$\begin{aligned}&\psi_2 \geq 0 \\ \Leftrightarrow \quad &(\lambda_{(1)} + \lambda_{(2)} + f_2)d \geq \lambda^2_{(1)} + \lambda_{(2)}(\lambda_{(1)} + \lambda_{(2)}) \geq 0 \\ \Rightarrow \quad &(\lambda_{(1)} + \lambda_{(2)}) + f_2 \geq 0.\end{aligned}$$

Notice that $F_2^{(2)} = \Lambda_2 + f_2 \geq 0$ implies $F_i^{(2)} = \Lambda_2 + f_i \geq 0$ for all $i \geq 0$. Finally, for $i \geq 3$, we can apply Equation (A.3)

$$\psi_i = \lambda_{(2)}^2 \psi_{i-2} + \lambda_{(1)} \Lambda_2 \psi_{i-2}^{(1)} + F_i^{(2)}(-g_{i-1}) \tag{A.7}$$

For ψ_i being nonnegative by Equation (A.7), we must require that $\psi_i^{(1)} \geq 0$ for all i, which is the case iff $\psi_2^{(1)} \geq 0$. Given that $\psi_1 \geq 0$ and by assuming that $\psi_2 \geq 0$, it follows from the recursion that $\psi_i \geq 0$ for all i.

"$\Rightarrow$"

If $\psi_i \geq 0$ for all i, then all assumptions are trivially satisfied. $\square$

Before we prove the next theorem, we need to establish the rate of convergence of the g_j coefficients. Since for the hypergeometric function it holds that

$$H(-d, 1; 1; L) = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(1+j)} L^j,$$

where $\Gamma(\cdot)$ is the gamma function which is defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ for $x > 0$, $\Gamma(x) = \infty$ for $x = 0$ and $\Gamma(x) = x^{-1}\Gamma(1+x)$ for $x < 0$, we have the representation

$$g_j = \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(1+j)} = O(j^{-d-1})$$

by applying Sterling's formula. Hence,

$$-g_j \lambda_i^{-j} \longrightarrow +\infty \quad \text{as } j \longrightarrow \infty,$$

which will be made use of in the subsequent proofs.

Proof of Theorem 2

Case 1: $1 > \lambda_1 \geq \lambda_2 > 0$.

"$\Leftarrow$"

Note that $\psi_i^{(1)}$ is identical with ψ_i from the FIGARCH(1, d, q) model, Case 1, with $\lambda_{(1)} = \lambda_1$.

1. We assume $\psi_1 \geq 0, \psi_2, \ldots, \psi_{q-1} \geq 0$. Note that $\psi_1 \geq 0$ implies $\psi_1^{(1)} \geq 0$.
2. (i) If either $\psi_q^{(1)} \geq 0$ and $F_{q+1} \geq 0$ or for k with $F_{k-1} < 0 \leq F_k$, we have that $\psi_{k-1}^{(1)} \geq 0$, by the same arguments as in the FIGARCH(1, d, q) model, Case 1, we can conclude that $\psi_i^{(1)} \geq 0$ for $i = q, \ldots$ and hence $\psi_{q-1} \geq 0$ (note, for $q = 1$ we require $\psi_1 \geq 0$ instead) implies $\psi_i \geq 0$ for all $i \geq q$.

(ii) If $\psi_{k-1}^{(1)} \leq 0$, then there exist $(\underline{k}, \overline{k})$ with $\underline{k} = \min\{j \mid \psi_j^{(1)} \leq 0, j = q, \ldots\}$, $\overline{k} = \min\{j \mid \psi_j^{(1)} \geq 0, j = k, \ldots\}$ (only if $q = 1$ do we have $q < \underline{k}$) s.t. $\psi_i^{(1)} \geq 0 \; \forall \; i \in \{q, \ldots, \underline{k} - 1\} \cup \{\overline{k}, \ldots\}$ and $\psi_i^{(1)} \leq 0 \; \forall \; i \in \{\underline{k}, \ldots, \overline{k} - 1\}$. $\overline{k}$ exists because we can write $\psi_{k+i}^{(1)}$ as

$$\psi_{k+i}^{(1)} = \lambda_{(1)}^{i+1} \psi_{k-1}^{(1)} + \sum_{j=0}^{i} F_{k+j}(-g_{k-q+j}) \lambda_{(1)}^{i-j},$$

and hence we must have that

$$0 \leq -\lambda_{(1)} \psi_{k-1}^{(1)} < \sum_{j=0}^{i} F_{k+j}(-g_{k-q+j}) \lambda_{(1)}^{-j}$$

for some i (the RHS is diverging) which gives $\overline{k} = k + i$. The existence of $\underline{k}$ is obvious, since $\psi_{k-1}^{(1)} \leq 0$. This implies $\psi_i \geq 0$ for $i = 1, \ldots, \underline{k} - 1$. Assuming that $\psi_{\overline{k}-1} \geq 0$ and starting with $i = \overline{k} - 1$, we derive recursively

$$\begin{aligned} \psi_i &= \lambda_2 \psi_{i-1} + \psi_i^{(1)} \geq 0 \\ &\Leftrightarrow \lambda_2 \psi_{i-1} \geq -\psi_i^{(1)} \geq 0 \\ &\Leftrightarrow \psi_{i-1} \geq 0 \end{aligned}$$

which implies that $\psi_i \geq 0$ with $i \in \{\underline{k}, \ldots, \overline{k} - 2\}$. Finally, $\psi_i \geq 0$ for $i \geq \overline{k}$, since $\psi_{\overline{k}-1} \geq 0$ and $\psi_i^{(1)} \geq 0 \; \forall \; i \geq \overline{k}$.

"$\Rightarrow$"

$\psi_1 \geq 0$ and $\psi_{\overline{k}-1} \geq 0$ are trivially satisfied. For k with $F_{k-1} < 0 \leq F_k$, we have that either $\psi_{k-1}^{(1)} \geq 0$ or $\psi_{k-1}^{(1)} \leq 0$, but in the latter case, there exists a $\overline{k}$ s.t. $\psi_{\overline{k}}^{(1)} \geq 0$ as shown above.

Case 2: $1 > \lambda_1 > 0 > \lambda_2 > -1$ and $\lambda_1 \geq |\lambda_2|$.

"$\Leftarrow$"

1. We assume $\psi_1 \geq 0, \psi_2, \ldots, \psi_{q-1} \geq 0$.
2. Set $\lambda_{(2)} = \lambda_2$ and $\lambda_{(1)} = \lambda_1$. Similar as in the FIGARCH(2, d, 0) model, we obtain for $i \geq q + 1$

$$\begin{aligned} \psi_i &= \lambda_{(2)} \psi_{i-1} + \psi_i^{(1)} \\ &= (\lambda_{(1)} + \lambda_{(2)}) \psi_{i-1} - \lambda_{(1)} \lambda_{(2)} \psi_{i-2} + F_i(-g_{i-q}) \end{aligned}$$

(i) Either $\psi_q \geq 0$ and $F_{q+1} \geq 0$ which implies $\psi_i \geq 0$ for all $i \geq q$ or

(ii) since $\lambda_{(1)} + \lambda_{(2)} \geq 0$, $\lambda_{(1)} \cdot \lambda_{(2)} < 0$, $g_i < 0$ and there exists a $k \geq q+2$ such that $F_{k-1} < 0 \leq F_k$, it suffices to assume that $\psi_1, \ldots, \psi_{k-1} \geq 0$ to ensure that $\psi_i \geq 0$ for all i. "⇒" $\psi_1, \ldots, \psi_{k-1} \geq 0$ is trivially satisfied, and the existence of k follows from $\lambda_{(1)} < 1$.

Case 3: $1 > \lambda_1 > 0 > \lambda_2 > -1$ and $\lambda_1 < |\lambda_2|$.

"⇐"

Note that $\psi_i^{(1)}$ is identical with ψ_i from the FIGARCH(1, d, q) model, Case 2, with $\lambda_{(1)} = \lambda_2$.

1. We assume $\psi_1 \geq 0, \psi_2, \ldots, \psi_{q-1} \geq 0$.
2. (i) If either $\psi_q^{(1)} \geq 0$, $\psi_{q+1}^{(1)} \geq 0$, and $F_{q+2}^{(1)} \geq 0$ or for k with $F_{k-1} < 0 \leq F_k^{(1)}$, we have that $\psi_{k-1}^{(1)} \geq 0$ and $\psi_{k-2}^{(1)} \geq 0$, by the same arguments as in the FIGARCH(1, d, q) model, Case 2, we can conclude that $\psi_i^{(1)} \geq 0$ for $i = q, \ldots$ and hence $\psi_{q-1} \geq 0$ (note, for $q = 1$ we require $\psi_1 \geq 0$ instead) implies $\psi_i \geq 0$ for all $i \geq q$.

   (ii) If $\psi_{k-1}^{(1)} \leq 0$ and/or $\psi_{k-2}^{(1)} \leq 0$, then there exists $\bar{k}$ with $q < k < \bar{k}$ s.t. $\psi_i^{(1)} \geq 0 \; \forall \; i \geq \bar{k}$. $\bar{k}$ exists because we can write $\psi_{k+2i}^{(1)}$ and $\psi_{k+1+2s}^{(1)}$ as

$$\psi_{k+2i}^{(1)} = \lambda_{(1)}^{2i+2}\psi_{k-2}^{(1)} + \sum_{j=0}^{i} F_{k+2j}^{(1)}(-g_{k-q+2j-1})\lambda_{(1)}^{2(i-j)}$$

$$\psi_{k+1+2s}^{(1)} = \lambda_{(1)}^{2s+2}\psi_{k-1}^{(1)} + \sum_{j=0}^{s} F_{k+1+2j}^{(1)}(-g_{k-q+2j})\lambda_{(1)}^{2(s-j)}$$

   and hence we must have that

$$0 \leq -\lambda_{(1)}^{2}\psi_{k-2}^{(1)} < \sum_{j=0}^{i} F_{k+2j}^{(1)}(-g_{k-q+2j-1})\lambda_{(1)}^{-2j}$$

$$0 \leq -\lambda_{(1)}^{2}\psi_{k-1}^{(1)} < \sum_{j=0}^{s} F_{k+1+2j}^{(1)}(-g_{k-q+2j})\lambda_{(1)}^{-2j}$$

   for some i, s (the RHS is diverging) which gives $\bar{k} = k + 2 \cdot \max\{i,s\}$. By assuming that $\psi_{q+1}, \ldots, \psi_{\bar{k}-1} \geq 0$, the non-negativity of ψ_i for all i follows directly.

"⇒"

$\psi_1 \geq 0$ and $\psi_{\bar{k}-1} \geq 0$ are trivially satisfied. For k with $F_{k-1}^{(1)} < 0 \leq F_k^{(1)}$, we either have that $\psi_{k-1}^{(1)} \geq 0$ and $\psi_{k-2}^{(1)} \geq 0$ or $\psi_{k-1}^{(1)} \leq 0$ and/or $\psi_{k-2}^{(1)} \leq 0$, but in the latter case, there exists a $\bar{k}$ s.t. $\psi_{\bar{k}}^{(1)} \geq 0$ as shown above.

Case 4: $0 > \lambda_1 \geq \lambda_2 > -1$, $\lambda_1 + \lambda_2 > -1$
"$\Leftarrow$"

Set $\lambda_{(1)} = \lambda_1$ and $\lambda_{(2)} = \lambda_2$. As in Case 2, we can write ψ_i as

$$\psi_i = \lambda_{(2)}\psi_{i-1} + \psi_i^{(1)}$$

with $\psi_i^{(1)}$ coming from the FIGARCH(1, d, q) model. But now, obviously we need to require $\psi_i^{(1)} \geq 0$ for all i. Using Equation (A.3), we obtain for $i \geq q+2$

$$\psi_i = \lambda_{(2)}^2\psi_{i-2} + \lambda_{(1)}\Lambda_2\psi_{i-2}^{(1)} + F_i^{(2)}(-g_{i-q-1}).$$

We know that there exists a k with $F_{k-1}^{(2)} < 0 \leq F_k^{(2)}$. Hence, assuming that $\psi_1, \ldots, \psi_{k-1} \geq 0$ implies $\psi_i \geq 0$ for all $i \geq k$.
"$\Rightarrow$"

$\psi_1, \ldots, \psi_{k-1} \geq 0$ and the existence of k are trivially satisfied. Moreover, $\psi_i^{(2)} > 0$ implies $\psi_i^{(1)} > 0$ for all i.

Case 5: $0 > \lambda_1 \geq \lambda_2 > -1$, $\lambda_1 + \lambda_2 < -1$

"$\Leftarrow$"
Choose $\lambda_{(1)}$ and $\lambda_{(2)}$ such that $\frac{\lambda_{(1)}}{\lambda_{(2)}} < 1$. Then by Equation (A.4)

$$\begin{aligned}
\psi_{i+2m}^{(2)} &= \lambda_{(2)}^{2m}\psi_i^{(2)} + \sum_{j=1}^{m}\lambda_{(2)}^{2(m-j)}\lambda_{(1)}\Lambda_2\psi_{i+2j-2}^{(1)} + \sum_{j=1}^{m}\lambda_{(2)}^{2(m-j)}F_{i+2j}^{(2)}(-g_{i-q+2j-1}) \\
&= \lambda_{(2)}^{2m}\psi_i^{(2)} + \sum_{j=1}^{m}\lambda_{(2)}^{2(m-j)}\lambda_{(1)}\Lambda_2\left[\lambda_{(1)}^{2j-2}\psi_i^{(1)} + \sum_{l=1}^{j}\lambda_{(1)}^{2(i-l)}F_{i+2l}^{(1)}(-g_{i-q+2l-1})\right] \\
&\qquad + \sum_{j=1}^{m}\lambda_{(2)}^{2(m-j)}F_{i+2j}^{(2)}(-g_{i-q+2j-1}) \\
&= \lambda_{(2)}^{2m}\psi_i^{(2)} + \lambda_{(2)}^{2m}\lambda_{(1)}^{-1}\Lambda_2\psi_i^{(1)}\sum_{j=1}^{m}\left(\frac{\lambda_{(1)}}{\lambda_{(2)}}\right)^{2j} + \sum_{j=1}^{m}\lambda_{(2)}^{2(m-j)}S_{j,i}.
\end{aligned} \tag{A.8}$$

Now, choose k such that $F_{k-1}^{(1)} < 0 \leq F_k^{(1)}$ and observe that

$$0 \leq \sum_{l=1}^{\infty}\lambda_{(1)}^{2(k-l)}F_{k+2l}^{(1)}(-g_{k-q+2l-1}) < \infty.$$

Hence, if there exists $\bar{k}$ with $S_{\bar{k},k} > 0$, then $S_{j,k} > 0$ for all $j > \bar{k}$ by monotonicity. Therefore, if $\psi_1, \ldots, \psi_{\bar{k}} \geq 0$, it follows from Equation (A.8) that $\psi_i \geq 0$ for all i.
"$\Rightarrow$"

$\psi_1, \ldots, \psi_{\bar{k}} \geq 0$ is trivially satisfied. If $\bar{k}$ does not exist, it follows from Equation (A.8) that ψ_{i+2m} will become negative for some m. As in Case 4, $\psi_i^{(2)} > 0$ implies $\psi_i^{(1)} > 0$ for all i.

Proof of Theorem 3

1. If $p = s$ and there is a λ_i s.t. $(\lambda_i, d, \phi_1, \ldots, \phi_q)$ satisfy the conditions for the FIGARCH(1, d, q) model, it is immediately clear from the recursion derived in Lemma 1 that $\psi_1 \geq 0$ is sufficient for $\psi_i \geq 0$ for all i. □

2. Let $p > s$. First, observe that $\psi_1 \geq 0$ implies $\psi_1^{(1)}, \ldots, \psi_1^{(p-1)} \geq 0$.
If $(\lambda_{(1)}, d, \phi_1, \ldots, \phi_q)$ satisfy the conditions for the FIGARCH(1, d, q) model, we can conclude that $\psi_i^{(1)} \geq 0$ for all i.
Since $\psi_i^{(1)} \geq 0$ for all i and $\lambda_{(2)} > 0$, we immediately obtain $\psi_i^{(2)} \geq 0$ for all i. $\psi_i^{(3)}$ can be written as

$$\psi_i^{(3)} = \lambda_{(3)}^2 \psi_{i-2}^{(3)} + \underbrace{(\lambda_{(2)} + \lambda_{(3)})}_{\geq 0} \psi_{i-1}^{(2)} + \psi_i^{(1)} \quad \text{for } i \geq 3.$$

Since $\psi_1^{(3)} = d + \phi_1 - (\lambda_{(1)} + \lambda_{(2)} + \lambda_{(3)}) \geq 0$, assuming $\psi_2^{(3)} \geq 0$ ensures that $\psi_i^{(3)} \geq 0$ for all i.
By the same arguments, we can show that $\psi_i^{(r)} \geq 0$ for all i follows from $\psi_i^{(r)} = \lambda_{(r)} \psi_{i-1}^{(r)} + \psi_i^{(r-1)}$ if $\psi_i^{(r-1)} \geq 0$ for all i and $r \leq 2(p-s)$ even and from $\psi_i^{(r)} = \lambda_{(r)}^2 \psi_{i-2}^{(r)} + (\lambda_{(r-1)} + \lambda_{(r)}) \psi_{i-1}^{(r-1)} + \psi_i^{(r-2)}$ by assuming $\psi_2^{(r)} \geq 0$ if $r \leq 1 + 2(p-s)$ odd.
For $r > 1 + 2(p-s)$, the recursion $\psi_i^{(r)} = \lambda_{(r)} \psi_{i-1}^{(r)} + \psi_i^{(r-1)}$ applies again. □

Proof of Theorem 4

Assume there exists an ordering with $\Lambda_{p_1} > -1$. Recall from Equation (6) that $F_i^{(r)} \to c > 0$ for all $r = 1, \ldots, p_1$.

We use the recursions (see Lemma 1)

$$\begin{aligned} \psi_i^{(1)} &= \lambda_{(1)} \psi_{i-1}^{(1)} + F_i(-g_{i-q}) \quad i > q \\ \psi_i^{(r)} &= \lambda_{(r)} \psi_{i-1}^{(r)} + \psi_i^{(r-1)} \quad 1 < r \leq p,\ i \geq 1, \text{ with } \quad \psi_0^{(r)} = -1 \end{aligned}$$

to deduce for $i > q + 1$

$$\psi_i^{(r)} = \sum_{k=0}^{r-1} \psi_{i-2}^{(r-k)} \lambda_{(r-k)} (\Lambda_r - \Lambda_{r-k-1}) + F_i^{(r)}(-g_{i-q-1}) \tag{A.9}$$

where $\Lambda_r - \Lambda_{r-k-1} < 0$ for all k as long as $r = 2, \ldots, p_1$. Repeated application of Equation (A.9) leads to

$$\psi_{i+2m}^{(r)} = \lambda_{(r)}^{2m}\psi_i^{(r)} + \sum_{k=1}^{r-1}\sum_{j=1}^{m}\lambda_{(r)}^{2(m-j)}\lambda_{(r-k)}(\Lambda_r - \Lambda_{r-k-1})\psi_{i+2j-2}^{(r-k)} + \sum_{j=1}^{m}\lambda_{(r)}^{2(m-j)}F_{i+2j}^{(r)}(-g_{i-q+2j-1}) \tag{A.10}$$

for $m = 1, 2, \ldots$ and $i > q+1$.

In what follows we show that for each r, there exists a k_r such that $\psi_j^{(r)} > 0$ for all $j \geq k_r$.

From the FIGARCH (1, d, q) model, we know that there exists a $k_1 > q$ such that $\psi_j^{(1)} > 0$ for all $j > k_1$.

This holds also in the case where $\lambda_1 > 0$, that is, $p_1 = 0$.

(i) $1 < r \leq p_1$
Assume that we have shown the existence of k_{r-1}. We know that $F_i^{(r)} = \Lambda_r F_{i-1} + F_i f_{i-q} \geq \tilde{c} > 0$ for all $i > \tilde{k}$. Then, for $i = \max\{k_{r-1}, \tilde{k}\}$ or $i = \max\{k_{r-1}, \tilde{k}\} + 1$, it is clear that every term on the right side of Equation (A.10) is positive except for $\psi_i^{(r)}$. If $\psi_i^{(r)} > 0$, it follows directly that $\psi_{i+2m}^{(r)} \geq 0$ for all m, and so we set $k_r = i$.

If $\psi_i^{(r)} < 0$, we see that $\psi_{i+2m}^{(r)} > 0$ is equivalent to

$$-\psi_i^{(r)} < \sum_{k=1}^{r-1}\sum_{j=1}^{m}\lambda_{(r)}^{-2j}\lambda_{(r-k)}(\Lambda_r - \Lambda_{r-k-1})\psi_{i+2j-2}^{(r-k)} + \sum_{j=1}^{m}\lambda_{(r)}^{-2j}F_{i+2j}^{(r)}(-g_{i+2j-q-1}) \tag{A.11}$$

Now the first sum on the right side is positive for all j, and the second sum tends to infinity as $F_{i+2j} \to c > 0$. Then it is obvious that there exists a $\bar{m}$ from which on this inequality is fulfilled. Then we set $k_r = \bar{m}$.

(ii) $p_1 < r \leq p_2$
Consider first the cases where $\lambda_{(r)} > 0$, that is, where $r - p_1$ is odd. Here we have that

$$\psi_i^{(r)} = \lambda_{(r)}\psi_{i-1}^{(r)} + \sum_{k=1}^{(r-p_1-1)/2}\left(\lambda_{(r-2k+1)}^2\psi_{i-2}^{(r-2k+1)} + (\lambda_{(r-2k+1)} + \lambda_{(r-2k)})\psi_{i-1}^{(r-2k)}\right) + \psi_i^{(p_1)}$$

and the iterated version

$$\psi_{i+m}^{(r)} = \lambda_{(r)}^{m}\psi_i^{(r)} + \sum_{j=1}^{m}\lambda_{(r)}^{m-j}\sum_{k=1}^{(r-p_1-1)/2}\big(\lambda_{(r-2k+1)}^2\psi_{i+j-2}^{(r-2k+1)} + (\lambda_{(r-2k+1)} + \lambda_{(r-2k)})\psi_{i+j-1}^{(r-2k)}\big) + \sum_{j=1}^{m}\lambda_{(r)}^{m-j}\psi_{i+j}^{(p_1)}$$

For $i = k_{r-1}$, every term on the RHS is positive except for $\psi_i^{(r)}$ which might be negative. If it is positive, we set $k_r = k_{r-1}$, if it is negative, we plug in Equation (A.10) for $\psi_{i+j}^{(p_1)}$ and obtain

$$\begin{aligned}\psi_{i+m}^{(r)} = {}& \lambda_{(r)}^{m}\psi_i^{(r)} + \sum_{j=1}^{m}\lambda_{(r)}^{m-j}\sum_{k=1}^{(r-p_1-1)/2}\big(\lambda_{(r-2k+1)}^2\psi_{i+j-2}^{(r-2k+1)} + (\lambda_{(r-2k+1)} \\ & + \lambda_{(r-2k)})\psi_{i+j-1}^{(r-2k)}\big) + \sum_{\substack{j=1\\ j \text{ odd}}}^{m}\lambda_{(r)}^{m-j}\psi_{i+j}^{(p_1)} + \sum_{\substack{j=1\\ j \text{ even}}}^{m}\lambda_{(r)}^{m-j}\lambda_{(p_1)}^{j}\psi_i^{(p_1)} \\ & + \sum_{\substack{j=1\\ j \text{ even}}}^{m}\lambda_{(r)}^{m-j}\sum_{k=1}^{p_1-1}\sum_{l=1}^{j/2}\lambda_{(p_1)}^{j-2l}\lambda_{(p_1-k)}(\Lambda_{p_1} - \Lambda_{p_1-k-1})\psi_{i+2l-2}^{(p_1-k)} \\ & + \sum_{\substack{j=1\\ j \text{ even}}}^{m}\lambda_{(r)}^{m-j}\sum_{l=1}^{j/2}\lambda_{(p_1)}^{j-2l}F_{i+2l}^{(p_1)}(-g_{i-q+2l-1}).\end{aligned}$$

Next we use the same argument as in Equation (A.11): dividing the whole equation by $\lambda_{(r)}^{m}$, we argue that the last sum on the right side diverges, as

$$\sum_{\substack{j=1\\ j \text{ even}}}^{m}\lambda_{(r)}^{-j}\sum_{l=1}^{j/2}\lambda_{(p_1)}^{j-2l}F_{i+2l}^{(p_1)}(-g_{i-q+2l-1}) \geq \sum_{\substack{j=1\\ j \text{ even}}}^{m}\lambda_{(r)}^{-j}F_{i+j}^{(p_1)}(-g_{i-q+j-1})$$

and the right sum tends to infinity by the usual argument. From this the existence of k_r follows.

Next consider $\lambda_{(r)} < 0$, that is, $r - p_1$ is even. We then have for $i - 2 > k_{r-1}$ that

$$\begin{aligned}\psi_i^{(r)} = {}& \lambda_{(r)}^2\psi_{i-2}^{(r)} + (\lambda_{(r)} + \lambda_{(r-1)})\psi_{i-1}^{(r-1)} + \sum_{k=1}^{(r-p_1)/2+1}\big(\lambda_{(r-2k)}^2\psi_{i-2}^{(r-2k)} + (\lambda_{(r-2k)} \\ & + \lambda_{(r-2k-1)})\psi_{i-1}^{(r-2k-1)}\big) + \psi_i^{(p_1)}.\end{aligned} \tag{A.12}$$

Every term on the RHS is positive except for $\psi_{i-2}^{(r)}$ which is possibly negative. If it is negative, iterating and inserting Equation (A.10) show the existence of k_r by the same arguments as in the case where $r - p_1$ is odd.

(iii) $p_2 < r \leq p$
Here we use the representation

$$\psi_i^{(r)} = \lambda_{(r)}\psi_{i-1}^{(r)} + \sum_{k=1}^{r-p_2+1} \lambda_{(r-k)}\psi_{i-1}^{(r-k)} + \psi_i^{(p_2)}$$

for $i > k_{r-1}$. If $\psi_{i-1}^{(r)}$ is positive, then every term on the right side is positive and we set $k_r = k_{r-1}$. If $\psi_{i-1}^{(r)}$ is negative, then we plug in Equation (A.12) for $\psi_i^{(p_2)}$ and then Equation (A.9) for $\psi_i^{(p_1)}$. Iteration and the same argument as in Equation (A.10) shows the existence of k_r.

Finally, set $k = k_p + 1$.

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