The correlation structure of some financial time series models

Menelaos Karanasos*

January 2004

Abstract

The purpose of this paper is to examine the correlation structure of mixed ARMA models, as discussed in Granger and Morris (1976). The technique we use to obtain the autocorrelations is based on the Wold representation of an ARMA model as it is given in Pandit (1973) or in Karanasos (2001). We give two examples to illustrate our general results: (i) two ARMA(2,2) processes with identical autoregressive polynomials and different moving average ones, and (ii) two ARMA(2,1) processes with different autoregressive and moving average polynomials.

Key Words: Autocorrelations, Mixed ARMA Models, Wold Representation, Aggregation.

We would like to thank M. Dietrich and M. Karanassou for their helpful comments and suggestions.

*Address for correspondence: Department of Economics and Related Studies, University of York, York, Y010 5DD; email: mk16@york.ac.uk, tel: +44 (0)1904 433799, fax: +44 (0)1904 433759.
1 Introduction

The purpose of this paper is to examine the correlation structure of mixed autoregressive and moving average (ARMA) time series models (introduced by Yule, 1921, 1927).

The theoretical autocovariance function (acf) of the ARMA model is an important tool in time series analysis. The calculation of the theoretical autocovariances is used (i) to estimate ARMA models with conventional exact maximum likelihood procedures (ii) to analyze the distribution of estimated ARMA parameters (Hannan, 1970) and (iii) to initialize simulations with ARMA models. The theoretical acf of the univariate ARMA($p, q$) model has been already derived in the literature. The autocovariances are obtained in terms of the roots of the autoregressive (AR) polynomial and the parameters of the moving average (MA) one.

Pandit (1973, pp. 100, 141) and Pandit and Wu\(^1\) (1983, pp. 105, 129-130), derived the acf of the ARMA($p, q$) model (when the roots of the AR polynomial are distinct) by using the infinite moving average (ima) or Wold representation\(^2\) of the aforementioned model. Nerlove et al. (1979, pp. 39, 78-85) used the canonical factorization (cf) of the autocovariance generating function (agf)\(^3\) to derive a general expression for the acf of the ARMA($p, q$) process (as an illustration they presented the acf of the MA($q$), AR($p$), and ARMA(1,1) models).

Zinde-Walsh (1988) obtained the acf of the ARMA($p, q$) model by using the standard (see for example, Priestley, 1981)\(^4\) spectral representation for the acf of a stationary ARMA process. She derived expressions for both cases of simple and multiple roots of the AR polynomial. Karanasos (1998) derived the acf of the ARMA($p, q$) model (when the roots of the AR polynomial are distinct) by expressing the ARMA($p, q$) model as an AR(1) process with an ARMA($p - 1, q$) error. Algorithms for computing the theoretical autocovariances for univariate ARMA processes have been suggested in McLeod (1975, 1977) and Tunnicliffe Wilson (1979).

This paper contributes to the above literature by deriving the acf of the sum of ARMA processes when the roots of the AR polynomials are distinct. Granger and Morris (1976) consider such mixed ARMA models. As most economic series are both aggregates and are measured with error it follows that these mixed models are often found in practice. To facilitate model identification, the results for the autocorrelations of the aggregate process can be applied so that properties of the observed data can be compared with the theoretical properties of the models.

We examine three distinct cases, depending on whether the processes have identical or different autoregressive parts and uncorrelated or correlated innovations. We obtain our results by using the ima representation of the ARMA($p, q$) model, as it is given in Pandit (1973) or in Karanasos (2001)\(^5\) of the ARMA($p, q$) model. The autocovariances are expressed in terms of the roots of the AR polynomials and the parameters of the MA ones. In the case of distinct roots Zinde-Walsh (1988) or Karanasos (1998) results are special cases of our Theorem 1 (see Proposition 1).

---

\(^1\)Pandit and Wu (1983) examine only the case where $q < p$.

\(^2\)They called the coefficient function in this expansion “Green’s function” (see Miller, 1968). What Pandit and Wu (1983) have called Green’s function is also referred to in the literature as weighting function (see Wiener, 1949, and Pugachev, 1957) or as $\psi$ weights (see Box and Jenkins, 1970).

For a proof of Wold’s theorem see Wold (1938, pp. 75-89), Hannan (1970, pp. 136-137), Anderson (1971, pp. 420-421), Sargent (1979, pp. 257-262) or Brockwell and Davis (1987, pp. 180-182); also see Hannan (1970, pp. 157-158) for the vector case. Wold’s theorem has also been discussed by, for example, Nerlove et al. (1979, pp. 30-36), Pandit and Wu (1983, pp. 87-89), and Reinsel (1993, p. 7).

The ima representation of the ARMA model has been discussed by, for example, Anderson (1976, p. 44), Pandit and Wu (1983, p. 108), Granger and Newbold (1986, pp. 25-26), Brockwell and Davis (1987, pp. 87-89), Wei (1989, p.56), Hamilton (1994, pp. 59-60), Gourieroux and Monfort (1997, pp. 160-162); also see Hannan (1970, pp. 157-158) for the vector case. The ima representation has also been discussed by, for example, Anderson (1976, p. 129), Granger and Newbold (1986, pp. 26-27), Brockwell and Davis (1987, pp. 102-103), Wei (1989, pp. 242-243), or Hamilton (1994, pp. 61-63); see also Brockwell and Davis (1987, p. 410) or Reinsel (1993, pp. 33-34), for the vector case.

\(^3\)The cf of the agf has also been discussed by, for example, Anderson (1976, p. 129), Granger and Newbold (1986, pp. 26-27), Brockwell and Davis (1987, pp. 102-103), Wei (1989, pp. 242-243), or Hamilton (1994, pp. 61-63); see also Brockwell and Davis (1987, p. 410) or Reinsel (1993, pp. 33-34), for the vector case.

\(^4\)For the important subject of spectral analysis see also the excellent books by Jenkins and Watts (1968), and Hannan (1967).

\(^5\)Karanasos (2001) has also used the ima representation in the context of the GARCH and GARCH in mean models.
2 Mixed ARMA models

Let \( y_t \) be an ARMA\((p, q)\) process given by

\[
\Phi(L)y_t = \mu + \Theta(L)\epsilon_t,
\]

with

\[
\Theta(L) = \sum_{k=0}^q \theta_k L^k,
\]

\[
\Phi(L) = -\sum_{k=0}^p \phi_k L^k = \prod_{k=1}^p (1 - \phi_k L), \quad (\phi_0 = -1),
\]

where \( \{\epsilon_t\} \) is a sequence of identical independent distributed random variables with mean zero and variance \( \sigma^2 \). In addition, \( \phi_k \) \( (k = 1, \ldots, p) \) denotes the inverse of the \( k \)th root of \( \Phi(L) \).

Assumption 1. All the roots of the autoregressive polynomial \( \Phi(L) \) lie outside the unit circle (stationarity condition).

Assumption 2. The polynomials \( \Phi(L) \) and \( \Theta(L) \) are left coprime. In other words the representation \( \Phi(L) \Theta(L) \) is irreducible.

In what follows we examine only the case where the roots of the autoregressive polynomial \( \Phi(L) \) are distinct \( (\phi_k \neq \phi_l \) for \( k, l = 1, \ldots, p \), and \( k \neq l \)). Our first proposition establishes the lag-\( m \) autocorrelation of \( y_t \), \( \rho_m(y_t) = \text{corr}(y_t, y_{t-m}), m \in \mathbb{N} \).

**Proposition 1.** Under Assumptions 1 and 2, the autocorrelation function of \( y_t \) is given by

\[
\rho_m(y_t) = \frac{\gamma_m}{\gamma_0}, \quad \gamma_m = \sum_{l=1}^p \zeta_{lm} \lambda_{l,\min(m,q)} \sigma^2, \quad (m \in \mathbb{N}),
\]

with

\[
\zeta_{lm} = \frac{\phi_l^{p-1+m} \prod_{k=1}^p (1 - \phi_l \phi_k) \prod_{k=1}^p (\phi_l - \phi_k)}{\prod_{k=1}^p (1 - \phi_l \phi_k) \prod_{k=1}^p (\phi_l - \phi_k)},
\]

and

\[
\lambda_{lm} = \sum_{k=0}^q \theta_k^2 + \sum_{d=1}^{m^*} \sum_{k=0}^{q-d} \theta_k \theta_{k+d}(\phi_l^d + \phi_l^{-d}) + \sum_{d=m^*+1}^q \sum_{k=0}^{q-d} \theta_k \theta_{k+d}(\phi_l^d + \phi_l^{-d-2m}),
\]

where \( m^* = \text{min}(m, q) \).

Proof. See Appendix A.

By considering the model generating the sum of two or more series, Granger and Morris (1976) have shown that the mixed ARMA model is the one most likely to occur. They mentioned that “two situations where series are added together are of particular interpretational importance. The first is where series are aggregated to form some total, and the second one is where the observed series is the sum of the true process plus an observational error. Most of the macroeconomic series, such as GNP, unemployment or exports are aggregates. Virtually any macroeconomic series, other than certain prices or interest rates, contain important observation errors.” So speaking the following analysis of the autocovariance function of the sum of ARMA processes can be a useful tool for the applied economist.

Next, we consider the situation where the stochastic process \( y_t \) is a simple linear aggregate of \( n \) processes \( y_{it} \), i.e.,

\[ \text{Zinde Victoria Walsh (1988) derived the above result (eq. 2) by using the standard (see for example, Priestley, 1981) spectral representation for the acf of a stationary ARMA process whereas Karanasos (1998) derived the same result by expressing the ARMA\((p,q)\) model as an AR(1) process with an ARMA\((p-1,q)\) error. Zinde-Walsh’s formula is general as it is not restricted to the case of distinct roots.} \]
\[ y_t = \sum_{i=1}^{n} y_{it}. \]  

Each individual process \( y_{it} \) is assumed to be a causal ARMA\((p,q)\) process satisfying

\[ \Phi(L)y_{it} = \mu_i + \Theta_i(L)\epsilon_{it}, \quad (i = 1, \ldots, n), \]

with

\[ \Theta_i(L) = \sum_{k=0}^{q_i} \theta_{ik} L^k, \]

where \( \Phi(L) \) is defined by (1). Further, we assume that the innovation vector \( \epsilon_t = (\epsilon_{1t}, \ldots, \epsilon_{mt})' \) is independent and identical distributed with mean vector zero and diagonal (positive definite) covariance matrix \( \Sigma \). That is, the \( ij \)th element of \( \Sigma \), for \( i = j \) is \( \sigma_{ii} \), and for \( i \neq j \) is zero.

Assumption 3. The polynomials \( \Phi(L) \) and \( \Theta_i(L), (i = 1, \ldots, n) \) are left coprime. In other words the representation \( \frac{\Phi(L)}{\Theta_i(L)} \) is irreducible.

Proposition 2. Let Assumptions 1 and 3 hold. The autocorrelation function of \( y_t \) is given by

\[ \rho_m(y_t) = \gamma_m, \quad \gamma_m = \sum_{i=1}^{n} \sum_{l=1}^{p} \zeta_{im} \lambda_{il,m} \sigma_{ii}, \quad (m \in \mathbb{N}), \]

with

\[ \lambda_{il,m} = \sum_{d=0}^{q_i} \epsilon_d^2 + \sum_{i=1}^{n} \sum_{l=1}^{p} \theta_{ir} \theta_{ir+d} (\phi_l^d + \phi_l^{-d}) + \sum_{i=1}^{n} \sum_{l=1}^{p} \theta_{ir} \theta_{ir+d} (\phi_l^d + \phi_l^{d-2m}), \]

where \( m^* = \min(m,q_i) \) and \( \zeta_{im} \) is defined in (2).

Proof. See Appendix A.

In the proposition that follows the assumption of independence is weakened to allow for contemporaneous correlation between the noise series.

Let \( y_t \) be a simple linear aggregate of \( n \) processes \( y_{it} \) defined by (3)-(4). In addition, we assume that the covariance matrix \( \Sigma \) is not diagonal. That is, the \( ij \)th \((i, j = 1, \ldots, n, \text{ and } i \neq j)\) element of \( \Sigma \) is \( \sigma_{ij} \neq 0 \).

Proposition 3. If Assumptions 1 and 3 hold, then the autocorrelation function of \( y_t \) is given by

\[ \rho_m(y_t) = \gamma_m, \quad \gamma_m = \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{p} \zeta_{ij,m} \lambda_{ij,lm} \sigma_{ij}, \quad (m \in \mathbb{N}), \]

with

\[ \lambda_{ij,lm} = \sum_{c=0}^{q_i} \sum_{d=0}^{q_j} \theta_{id} \theta_{jd} \phi_l^c + \sum_{c=1}^{m_i^*} \sum_{d=0}^{q_j} \theta_{jd} \theta_{ic} \phi_l^{-c} + \sum_{c=1}^{m_i^*} \sum_{d=0}^{q_j} \theta_{jd} \theta_{ic} \phi_l^{-2m}, \]

where \( q_i^* = \min(q_i, q_j - c) \), \( q_j^* = \min(q_j, q_i - c) \), \( m_i^* = \min(m, q_i) \) and \( \zeta_{im} \) is defined in (2).

Proof. See Appendix A.

A potential motivation for the derivation of the results in Proposition 3 is that the autocorrelations of the stochastic process \( y_t \) in (3) can be used to estimate the parameters in (4). The approach is to use the minimum distance estimator (MDE), which estimates the parameters by minimizing the Mahalanobis generalized distance of a vector of sample autocorrelations from the corresponding population autocorrelations (see Baillie and Chung, 2001).

As an illustration, consider two ARMA\((2,2)\) processes \( y_{1t} \) and \( y_{2t} \).
\[(1 - \phi_1 L)(1 - \phi_2 L)y_{1t} = \mu_1 + (1 + \theta_{11} L + \theta_{12} L^2)\epsilon_{1t}, \quad (7a)\]
\[(1 - \phi_1 L)(1 - \phi_2 L)y_{2t} = \mu_2 + (1 + \theta_{21} L + \theta_{22} L^2)\epsilon_{2t}, \quad (7b)\]
with
\[
\begin{pmatrix}
\varepsilon_{1t} \\
\varepsilon_{2t}
\end{pmatrix} \sim \text{i.i.d.}(0, \Sigma), \quad \Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{pmatrix}.
\]
Further, it is assumed that \(\phi_1 \neq \phi_2\) and \(|\phi_1|, |\phi_2| < 1\).

The cross-covariance between \(y_{1t}\) and \(y_{2t}\) is
\[
\text{cov}(y_{1t}, y_{2t}) = \sigma_{12} \times
\]
\[
\frac{\phi_1[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{21} + \theta_{11}\theta_{22} + \theta_{11} + \theta_{21}\theta_{12})\phi_1 + (\theta_{12} + \theta_{22})\phi_1^2]}{(1 - \phi_1 \phi_2)(1 - \phi_1^2)(\phi_1 - \phi_2)} +
\phi_2[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{21} + \theta_{11}\theta_{22} + \theta_{11} + \theta_{21}\theta_{12})\phi_2 + (\theta_{12} + \theta_{22})\phi_2^2]
\frac{(1 - \phi_1 \phi_2)(1 - \phi_2^2)(\phi_2 - \phi_1)}{(1 - \phi_1 \phi_2)(1 - \phi_2^2)(\phi_2 - \phi_1)}.
\]
Moreover, the cross-covariances between the two processes, at lag one, are given by
\[
\text{cov}(y_{1t}, y_{2,t-1}) = \sigma_{12} \times
\]
\[
\frac{\phi_1^2[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{21} + \theta_{11}\theta_{22} + \theta_{11} + \theta_{21}\theta_{12})\phi_1 + (\theta_{11} + \theta_{21}\theta_{12})\phi_1^{-1} + \theta_{22}\phi_1^2 + \theta_{12}\phi_1^0]}{(1 - \phi_1 \phi_2)(1 - \phi_1^2)(\phi_1 - \phi_2)} +
\phi_2^2[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{21} + \theta_{11}\theta_{22} + \theta_{11} + \theta_{21}\theta_{12})\phi_2 + (\theta_{11} + \theta_{21}\theta_{12})\phi_2^{-1} + \theta_{22}\phi_2^2 + \theta_{12}\phi_2^0]
\frac{(1 - \phi_1 \phi_2)(1 - \phi_2^2)(\phi_2 - \phi_1)}{(1 - \phi_1 \phi_2)(1 - \phi_2^2)(\phi_2 - \phi_1)}.
\]
\[
\text{cov}(y_{2t}, y_{1,t-1}) = \sigma_{12} \times
\]
\[
\frac{\phi_1^2[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{21} + \theta_{11}\theta_{22} + \theta_{11} + \theta_{21}\theta_{12})\phi_1 + (\theta_{21} + \theta_{11}\theta_{22})\phi_1^{-1} + \theta_{12}\phi_1^2 + \theta_{22}\phi_1^0]}{(1 - \phi_1 \phi_2)(1 - \phi_1^2)(\phi_1 - \phi_2)} +
\phi_2^2[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{21} + \theta_{11}\theta_{22} + \theta_{11} + \theta_{21}\theta_{12})\phi_2 + (\theta_{21} + \theta_{11}\theta_{22})\phi_2^{-1} + \theta_{22}\phi_2^2 + \theta_{12}\phi_2^0]
\frac{(1 - \phi_1 \phi_2)(1 - \phi_2^2)(\phi_2 - \phi_1)}{(1 - \phi_1 \phi_2)(1 - \phi_2^2)(\phi_2 - \phi_1)}.
\]
Finally, the cross-covariances between \(y_1\) and \(y_2\), at lag \(m\) \((m \geq 2)\), are
\[
\text{cov}(y_{1t}, y_{2,t-m}) = \sigma_{12} \times
\]
\[
\phi_1^{1+m}[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{21} + \theta_{11}\theta_{22})\phi_1 + (\theta_{11} + \theta_{21}\theta_{12})\phi_1^{-1} + \theta_{22}\phi_1^2 + \theta_{12}\phi_1^{-2}]
\frac{(1 - \phi_1 \phi_2)(1 - \phi_1^2)(\phi_1 - \phi_2)}{(1 - \phi_1 \phi_2)(1 - \phi_1^2)(\phi_1 - \phi_2)}
\]
\[
+ \phi_2^{1+m}[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{21} + \theta_{11}\theta_{22})\phi_2 + (\theta_{11} + \theta_{21}\theta_{12})\phi_2^{-1} + \theta_{22}\phi_2^2 + \theta_{12}\phi_2^{-2}]
\frac{(1 - \phi_1 \phi_2)(1 - \phi_2^2)(\phi_2 - \phi_1)}{(1 - \phi_1 \phi_2)(1 - \phi_2^2)(\phi_2 - \phi_1)}.
\]
\[
\text{cov}(y_{2t}, y_{1,t-m}) = \sigma_{12} \times \\
\left\{ \phi_1^{1+m}[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{11} + \theta_{21}\theta_{12})\phi_1 + (\theta_{21} + \theta_{11}\theta_{22})\phi_1^{-1} + \theta_{12}\phi_1^2 + \theta_{22}\phi_1^{-2}] \\
(1 - \phi_1\phi_2)(1 - \phi_2^2)(\phi_1 - \phi_2) \right. \\
+ \phi_2^{1+m}[1 + \theta_{11}\theta_{21} + \theta_{12}\theta_{22} + (\theta_{11} + \theta_{21}\theta_{12})\phi_2 + (\theta_{21} + \theta_{11}\theta_{22})\phi_2^{-1} + \theta_{12}\phi_2^2 + \theta_{22}\phi_2^{-2}] \right\}.
\]

Note that, for \( \theta_{11} = \theta_{21}, \theta_{12} = \theta_{22} \) and \( \epsilon_{1t} = \epsilon_{2t} \), the two processes are identical \((y_{1t} = y_{2t} = y_t)\) and the above expressions give the autocovariances of the \( y_t \) process. That is

\[
\text{cov}(y_{1t}, y_{2,t-m}) = \text{cov}(y_{2t}, y_{1,t-m}) = \text{cov}_m(y_t).
\]

In the following theorem the assumption of identical autoregressive parts is relaxed.

We consider the stochastic process \( y_t \) defined by (3) and we assume that each individual process \( y_{it} \) follows an ARMA\((p_i, q_i)\) process

\[
\Phi_i(L)y_{it} = \mu_i + \Theta_i(L)\epsilon_{it},
\]

with

\[
\Phi_i(L) = -\sum_{k=0}^{p_i} \phi_{ik}^L L^k = \prod_{k=1}^{p_i} (1 - \phi_{ik} L),
\]

where \( \Theta_i(L) \) is defined by (4). In addition, we assume that the innovation vector \( \epsilon_t = (\epsilon_{1t}, \ldots, \epsilon_{nt})' \) is independent identically distributed with mean vector zero and (positive definite) covariance matrix \( \Sigma \). The \( ij \)th \((i, j = 1, \ldots, n)\) element of \( \Sigma \), is denoted by \( \sigma_{ij} \). Finally, \( \phi_{ik} (k = 1, \ldots, p_i) \) denotes the inverse of the \( k \)th root of \( \Phi_i(L) \).

Assumption 4. All the roots of the autoregressive polynomials \( [\Phi_i(L)] \) lie outside the unit circle (stationarity conditions).

Assumption 5. The polynomials \( \Phi_i(L) \) and \( \Theta_i(L) \) \((i = 1, \ldots, n)\) are left coprime. In other words the representation \( \Phi_i(L) \) is irreducible.

In what follows we examine only the case where the roots of the autoregressive polynomial \( \Phi_i(L) \) are distinct \((\phi_{ik} \neq \phi_{il}, \text{ for } k \neq l)\).

**Theorem 1.** Under Assumptions 4 and 5, the autocorrelation function of \( y_t \) is

\[
\rho_m(y_t) = \frac{\gamma_m}{\gamma_0}, \quad \gamma_m = \sum_{j=1}^{n} \sum_{i=1}^{n} \left( \sum_{l=1}^{p_i} \zeta_{il,jm}\lambda_{ij,lm} + \sum_{k=1}^{p_j} \zeta_{jkm,ij}\lambda_{km,ij} \right)\sigma_{ij}, \quad (m \in \mathbb{N}),
\]

with

\[
\zeta_{il,jm} = \frac{\zeta_{il,m}}{\prod_{k=1}^{p_i} (1 - \phi_{il}\phi_{jk})},
\]

\[
\zeta_{il,m} = \frac{\phi_{il}^{p_i-1+m}}{\prod_{k=1}^{p_i} (\phi_{il} - \phi_{ik})}, \quad \text{for } k \neq l
\]

and
\[
\lambda_{ij,lm} = \sum_{c=0}^{q_i} \sum_{d=0}^{q_j'} \theta_{id}\phi_{j,d+c}\phi_{ij}^c + \sum_{c=1}^{m_i^*} \sum_{d=0}^{q_j'} \theta_{id}\phi_{j,d+c}\phi_{ij}^{c-1},
\]
\[
\lambda_{km,ij} = \sum_{c=m_i^*+1}^{q_i} \sum_{d=0}^{q_j'} \theta_{id}\phi_{j,d+c}\phi_{j,k}^{c-2m},
\]
where \(q_i', q_j'\) and \(m_i^*\) are defined by (6).

Proof. See Appendix B.

The derivation of the autocorrelations of mixed ARMA models and their comparison with the corresponding sample equivalents will help the investigator (a) to decide which is the most appropriate method of estimation (e.g., maximum likelihood estimation (MLE), minimum distant estimator (MDE) ) for a specific model, (b) to choose, for a given estimation technique, the model that best replicates certain stylized facts of the data and, (c) in conjunction with the various model selection criteria, to identify the optimal order of the chosen specification.

As an illustration, consider two ARMA(2,1) processes

\[
(1 - \phi_{11} L)(1 - \phi_{12} L)y_{1t} = (1 + \theta_{11} L)e_{1t}, \quad (10a)
\]
\[
(1 - \phi_{21} L)(1 - \phi_{22} L)y_{2t} = (1 + \theta_{21} L)e_{2t}, \quad (10b)
\]

where \(\mathbb{I}(\varepsilon_{1t}, \varepsilon_{2t}) \sim \text{i.i.d.}(0, \Sigma), \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}.\)

It is assumed that \(|\phi_{i1}|, |\phi_{i2}| < 1\) and \(\phi_{i1} \neq \phi_{i2} \quad (i = 1, 2).\)

The cross-covariance between \(y_{1t}\) and \(y_{2t}\) is

\[
cov(y_{1t}, y_{2t}) = \sigma_{12} \times \frac{\phi_{11}[1 + \theta_{11}\theta_{21} + \theta_{21}\phi_{11}]}{(\phi_{11} - \phi_{12})(1 - \phi_{11}\phi_{21})(1 - \phi_{11}\phi_{22})} + \frac{\phi_{12}[1 + \theta_{11}\theta_{21} + \theta_{21}\phi_{12}]}{(\phi_{12} - \phi_{11})(1 - \phi_{12}\phi_{21})(1 - \phi_{12}\phi_{22})} + \\
\frac{\phi_{21}^2\theta_{11}}{(\phi_{21} - \phi_{22})(1 - \phi_{21}\phi_{11})(1 - \phi_{21}\phi_{12})} + \frac{\phi_{22}^2\theta_{11}}{(\phi_{22} - \phi_{21})(1 - \phi_{22}\phi_{11})(1 - \phi_{22}\phi_{12})}.\]

Moreover, the covariances between the two processes, at lag \(m\) \((m \geq 1)\), are given by

\[
cov(y_{1t}, y_{2t-m}) = \frac{\phi_{11}^{1+m}[1 + \theta_{11}\theta_{21} + \theta_{21}\phi_{11} + \theta_{11}\phi_{11}^{1}]}{(\phi_{11} - \phi_{12})(1 - \phi_{11}\phi_{21})(1 - \phi_{11}\phi_{22})} + \\
\frac{\phi_{12}^{1+m}[1 + \theta_{11}\theta_{21} + \theta_{21}\phi_{12} + \theta_{11}\phi_{12}^{1}]}{(\phi_{12} - \phi_{11})(1 - \phi_{12}\phi_{21})(1 - \phi_{12}\phi_{22})} \sigma_{12},
\]

and
\[ \text{cov}(y_{2t}, y_{1,t-m}) = \left\{ \frac{\varphi_{21}^{1+m}[1 + \theta_{11}\theta_{21} + \theta_{11}\phi_{21} + \theta_{21}\phi_{21}^{-1}]}{(\phi_{21} - \phi_{22})(1 - \phi_{21}\phi_{11})(1 - \phi_{21}\phi_{12})} + \right. \\
+ \left. \frac{\varphi_{22}^{1+m}[1 + \theta_{11}\theta_{21} + \theta_{11}\phi_{22} + \theta_{21}\phi_{22}^{-1}]}{(\phi_{22} - \phi_{21})(1 - \phi_{22}\phi_{11})(1 - \phi_{22}\phi_{12})} \right\} \sigma_{12}. \]

Note that, when \( \phi_{11} = \phi_{21} = \phi_{1}, \phi_{12} = \phi_{2} = \phi_{22} = \phi_{2} \) and \( \theta_{12} = \theta_{22} = 0 \) in equations (10) and (7), respectively, the corresponding expressions for the cross-covariance are equal.

Finally, we should mention that the techniques in this paper can be employed to calculate the cross-correlations between the cross-products of the residuals in multivariate nonlinear time series models. As economic and financial variables are inter-related, generalization of univariate models to the multivariate set-up is quite natural, in particular for the GARCH models. These nonlinear processes have been widely used in empirical applications. More specifically, they have been used by: Bollerslev et al. (1988) for their analysis of returns, bills, bonds and stocks; Baillie and Bollerslev (1990) to model risk premia in forward foreign exchange rate markets; Baillie and Myers (1991) to estimate optimal hedge ratios in commodity markets; Kroner and Claessens (1991) to analyze the optimal currency composition of external debt; Ng (1991) to test the CAPM; McCurby and Morgan (1991) to test the uncovered interest rate parity; Grier and Perry (2000) to examine the effects of real and nominal uncertainty on inflation and output growth. The extensive use of these processes in modelling conditional volatility in high frequency financial assets demonstrates the popularity of the various multivariate GARCH models (see, for example, the excellent surveys by Shephard, 1996, and Pagan, 1996).

3 Concluding remarks

The purpose of this paper was to examine the correlation structure of mixed ARMA models. In Section 2 we presented the autocorrelation function of the sum of \( n \) ARMA\((p_{i}, q_{i})\) processes using the infinite moving average representation of an ARMA model. The autocorrelations were expressed in terms of the roots of the AR polynomials and the parameters of the MA polynomials. In this paper we examined only the case of distinct roots of the autoregressive polynomials-the case of equal roots is left for future research. The ima representation technique can also be applied to obtain the correlation structure of non linear multivariate time series models, e.g. GARCH and Markov Switching models.

REFERENCES


**Appendix A**

**PROOF OF PROPOSITION 1**

The impulse response function (irf) of the ARMA($p, q$) process $y_t$ (eq. 1) is given by

$$y_t = \mu + \sum_{r=0}^{\infty} e_r \epsilon_{t-r},$$  \hspace{1cm} (A.1)

where

$$e_r = \sum_{l=1}^{p} \sum_{k=0}^{\min(r,q)} \zeta_{l,r-k} \theta_k,$$

and

$$\zeta_{lm} = \frac{\phi_l^{p-1+m}}{\prod_{k=1 \atop k \neq l} (\phi_l - \phi_k)}$$

(see Pandit, 1973 or Karanasos, 2001).

Multiplying $(y_t - \mu)$ by $(y_{t-m} - \mu)$ $(m \in \mathbb{N})$ and taking expected values gives the lag-$m$ autocovariance of $y_t$

$$\text{cov}_m(y_t) = \sum_{r=0}^{\infty} e_r e_{r+m} \sigma^2.$$  \hspace{1cm} (A.2)

By direct computation, we have

$$\text{cov}_m(y_t) = \sum_{l=1}^{p} \tilde{\zeta}_{lm} s_{l0} \lambda_{lm} \cdot \sigma^2,$$

with

$$s_{l0} = \sum_{k=1}^{p} \frac{\tilde{\zeta}_{k0}}{(1 - \phi_l \phi_k)},$$

and

$$\lambda_{lm} = \sum_{k=0}^{q} \theta_k^2 + \sum_{d=1}^{m^*} \sum_{k=0}^{q-d} \theta_k \theta_{k+d} (\phi_l^d + \phi_l^{-d}) + \sum_{d=m^*+1}^{q} \sum_{k=0}^{q-d} \theta_k \theta_{k+d} (\phi_l^d + \phi_l^{-d-m})$$

where $m^* = \min(m, q)$ and $\tilde{\zeta}_{lm}$ is given by (A.1).

It is readily verified that

$$s_{l0} = \frac{1}{\prod_{k=1}^{p} (1 - \phi_l \phi_k)}.$$

Therefore, we may write
\[
\text{cov}_m(y_t) = \sum_{l=1}^{p} \zeta_{lm} \lambda_{l, \min(m,q)} \sigma^2_i, \quad (A.3)
\]

with

\[
\zeta_{lm} = \frac{\phi_l^{p-1+m} \prod_{k=1}^{p} (1 - \phi_l \phi_k) \prod_{k \neq l} (\phi_l - \phi_k)}{\prod_{k=1}^{p} (1 - \phi_l \phi_k) \prod_{k \neq l} (\phi_l - \phi_k)},
\]

where \(\lambda_{lm}\) is given by (A.2).

**PROOF OF PROPOSITION 2**

From (3) and the results in Proposition 1, we have

\[
\text{cov}_m(y_t) = \sum_{i=1}^{n} \text{cov}_m(y_{it}) = \sum_{i=1}^{n} \sum_{l=1}^{p} \zeta_{lm} \lambda_{l, \min(m,q)} \sigma^2_{ii},
\]

where

\[
\lambda_{l,t,m} = \sum_{d=0}^{q_i} \theta_{id}^2 + \sum_{d=1}^{m} \sum_{d'=1}^{q_i-d} \theta_{id} \theta_{id'+d} (\phi_l^d + \phi_l^{d-2m}),
\]

and \(\zeta_{lm}\) is given by (A.3).

**PROOF OF PROPOSITION 3**

The irf of the ARMA\((p,q_i)\) process \(y_{it}\) (eq. 4) is given by

\[
y_{it} = \mu_i + \sum_{r=0}^{\infty} e_{ir} \epsilon_{it-r},
\]

with

\[
e_{ir} = \sum_{l=1}^{p} \sum_{k=0}^{\min(r,q_i)} \tilde{\zeta}_{l,r-k} \theta_{ik},
\]

where \(\tilde{\zeta}_{lm}\) is given by (A.1).

Multiplying \((y_{it} - \mu_i)\) by \((y_{j,t-m} - \mu_j)\) and taking expected values, under the assumption that the elements of \(\Sigma\) on the off-diagonal positions are not zero, gives the cross-covariances between \(y_{it}\) and \(y_{jt}\), at lag \(m\) \((m \in N)\)

\[
\text{cov}(y_{it}, y_{jt-m}) = \sum_{r=0}^{\infty} e_{jr} e_{i,r+m} \sigma_{ij}, \quad (A.4)
\]

Straightforward algebra establishes that

\[
\text{cov}(y_{it}, y_{jt-m}) = \sum_{l=1}^{p} \tilde{\zeta}_{lm} s_{l0} \lambda_{l,j,m} \sigma_{ij},
\]

with

\[
\tilde{\zeta}_{lm} = \frac{\phi_l^{p-1+m} \prod_{k=1}^{p} (1 - \phi_l \phi_k) \prod_{k \neq l} (\phi_l - \phi_k)}{\prod_{k=1}^{p} (1 - \phi_l \phi_k) \prod_{k \neq l} (\phi_l - \phi_k)},
\]
\[ \lambda_{ij,lm} = \sum_{c=0}^{q_j} \sum_{d=0}^{q'_j} \theta_{id}\phi_{j,d} + \sum_{c=1}^{m} \sum_{d=0}^{q'_j} \theta_{jd}\phi_{i,d} + \sum_{c=m+1}^{q_j} \sum_{d=0}^{q'_j} \theta_{jd}\phi_{i,d}^{-2m}; \]

where \( s_{l0} \) is given by (A.2).

Recall that \( s_{l0} = \frac{1}{\prod_{k=1}^{p_i} (1-\phi_k\phi_k)} \). Then we may write

\[ \text{cov}(y_{it}, y_{jt-m}) = \sum_{l=1}^{p} \zeta_{lm} \lambda_{ij,lm} \sigma_{ij}, \]

where \( \zeta_{lm} \) is given by (A.3).

Using the preceding equation, (3) and (4), we obtain the lag-\( m \) autocovariance of \( y_t \)

\[ \text{cov}_m(y_t) = \sum_{j=1}^{n} \sum_{i=1}^{n} \text{cov}(y_{it}, y_{jt-m}) = \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{p} \zeta_{lm} \lambda_{ij,lm} \sigma_{ij}. \]

Appendix B

PROOF OF THEOREM 1

The irf of the ARMA(\( p_i, q_i \)) process \( y_{it} \) (eq. 8) is given by

\[ y_{it} = \mu_i + \sum_{r=0}^{\infty} e_{ir} e_{it-r}, \quad \text{eq. (B.1)} \]

where

\[ e_{ir} = \sum_{l=1}^{p_i} e_{i+r\theta_{il}}, \quad \text{eq. (B.1a)} \]

and

\[ \zeta_{il,m} = \frac{\phi_{il}^{p_i-1+m}}{\prod_{k=1, k\neq l}^{p_i} (\phi_{il} - \phi_{ik})}. \]

The same argument leading to (A.4) can be used to establish that

\[ \text{cov}(y_{it}, y_{jt-m}) = \sum_{r=0}^{\infty} e_{ir} e_{i+r+m} \sigma_{ij}. \]

Using (B.1a) this can be expressed as

\[ \text{cov}(y_{it}, y_{jt-m}) = \left( \sum_{l=1}^{p_i} \zeta_{il,m} s_{j0,il} \lambda_{ij,lm} + \sum_{k=1}^{p_j} \zeta_{jk,m} s_{j0,jk} \lambda_{km,ij} \right) \sigma_{ij}, \quad \text{eq. (B.2)} \]

with

\[ s_{j0,il} = \sum_{k=1}^{p_j} \frac{\zeta_{jk,0}}{(1-\phi_{il}\phi_{jk})}, \]

and
\[ \lambda_{ij,lm} = \sum_{c=0}^{q_i} \sum_{d=0}^{q_j'} \theta_{id}\theta_{j,d+c}\phi_{il}^{c} + \sum_{c=1}^{m_i^*} \sum_{d=0}^{q_j'} \theta_{jd}\theta_{i,d+c}\phi_{il}^{c}, \]
\[ \lambda_{km,ij} = \sum_{c=m_i^*+1}^{q_i} \sum_{d=0}^{q_j'} \theta_{jd}\theta_{id+c}\phi_{jk}^{c-2m}, \]

where \( q_i' = \min(q_i, q_j - c) \), \( q_j' = \min(q_j, q_i - c) \), \( m_i^* = \min(m, q_i) \) and \( \zeta_{il,m} \) is given by (B.1a). Further, it can be shown that

\[ s_{lj,il} = \frac{1}{\prod_{k=1}^{p_j} (1 - \phi_{il}\phi_{jk})}. \]

Therefore, (B.2) becomes

\[ \text{cov}(y_{it}, y_{jt-m}) = \left( \sum_{l=1}^{p_i} \zeta_{il,jm}\lambda_{ij,lm} + \sum_{k=1}^{p_j} \zeta_{jk,im}\lambda_{km,ij} \right) \sigma_{ij}, \]

where

\[ \zeta_{il,jm} = \frac{\zeta_{il,m}}{\prod_{k=1}^{p_j} (1 - \phi_{il}\phi_{jk})}. \]

Thus, the covariance of \( y_t \), at lag \( m \), is given by

\[ \text{cov}_m(y_t) = \sum_{j=1}^{n} \sum_{i=1}^{n} \text{cov}(y_{it}, y_{jt-m}) = \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{p_i} \zeta_{il,jm}\lambda_{ij,lm} + \sum_{k=1}^{p_j} \zeta_{jk,im}\lambda_{km,ij} \sigma_{ij}. \]