Problem Set

1. Consider a certain economy where $f(t, T) > r_T$. Show that an arbitrage opportunity can exist through trading at time t in T and $(T+1)$-maturity zero coupon bonds.

2. Consider two traders, A and B, who trade zeros according to their equilibrium beliefs: trader A uses the "mechanical" rule based on the Unbiased Expectations Hypothesis (UEH), whereas trader B uses the more "economics" rule based on the Return-to-Maturity Expectations Hyppothesis (RTM). Under which condition traders A and B will agree in their equilibrium prices?

3. In the context of the zero coupon bond market, illustrate and explain the internal inconsistency in the Return-to-Maturity (RTM) expectations theory.

4. Show how the eqilibrium prices of discount bonds are derived in an uncertain economy by using the Unbiased, Return-to-Maturity, Yield-to-Maturity, and Local Expectations Hypotheses.

5. Assume that the Local Expectations Hypothesis describes equilibrium in the zero coupon bond market. However, there exist traders who price the bonds according to the Return-to-Maturity and Yield-to-Maturity Expectations Hypotheses. Show that price premiums arise in each case.

6. Solve the following exercises from Brealey & Myers, chapter 23:
ex. 2a-2f, 3, 4, 5, 11, 12, and 13 from "Questions and Problems".

"The (yield-to-maturity/spot-rate) formula discounts all cash flows from one bond at the same rate even though they occur at different points in time." "The (yield-to-maturity/spot-rate) formula discounts all cash flows received at the same point in time at the same rate even though the cash flows may come from different bonds."

3. Use Table 23-1 to check your answers to the following:
   (*a*) If interest rates rise, do bond prices rise or fall?
   (*b*) If the bond yield is greater than the coupon, is the price of the bond greater or less than 100?
   (*c*) If the price of a bond exceeds 100, is the yield greater or less than the coupon?
   (*d*) Do high-coupon bonds sell at higher or lower prices than low-coupon bonds?
4. Use Table 23-1 to answer the following questions:
   (*a*) What is the yield to maturity on a 7 percent, 8-year bond selling at 74%?
   (*b*) What is the approximate price of a 7 percent, 9-year bond yielding 10 percent?
   (*c*) An 8 percent, 12-year bond yields 14 percent. If the yield remains unchanged, what will be its price 2 years hence? What will the price be if the yield falls to 10 percent?
5. (*a*) Suppose that the 1-year spot rate of interest at time 0 is 1 percent and the 2-year spot rate is 3 percent. What is the forward rate of interest for year 2?
   (*b*) What does the expectations theory of term structure say about the relationship between this forward rate and the 1-year spot rate at time 1?
   (*c*) Over a very long period of time, the term structure in the United States has been, on average, upward-sloping. Is this evidence for or against the expectations theory?
   (*d*) What does the liquidity-preference theory say about the relationship between the forward rate and the 1-year spot rate at time 1?
   (*e*) If the liquidity-preference theory is a good approximation and you have to meet long-term liabilities (college tuition for your children, for example), is it safer to invest in long-term or short-term bonds? Assume inflation is predictable.
   (*f*) If the inflation-premium theory is a good approximation and you have to meet long-term real liabilities, is it safer to invest in long-term or short-term bonds?
   (*g*) What does the inflation-premium theory say about the relationship between the forward rate and the 1-year spot rate at time 1?
6. (*a*) State the four Moody's ratings which are generally known as "investment-grade" ratings.
   (*b*) Other things equal, would you expect the yield to maturity on a corporate bond to increase or decrease with:
      (i) The company's business risk?
      (ii) The expected rate of inflation?
      (iii) The risk-free rate of interest?
      (iv) The degree of leverage?

*7. (*a*) How in principle would you calculate the value of a government loan guarantee?
   (*b*) The difference between the price of a government bond and a simple corporate bond is equal to the value of an option. What is this option and what is its exercise price?

8. True or false? Explain.
   (*a*) Longer maturity bonds necessarily have longer durations.
   (*b*) The longer a bond's duration, the lower is its volatility.
   (*c*) Other things equal, the lower the bond coupon, the higher its volatility.
   (*d*) If interest rates rise, bond durations rise also.
9. Calculate the durations and volatilities of securities A, B, and C. Their cash flows are shown below. The interest rate is 8 percent.

	Period 1	Period 2	Period 3
A	40	40	40
B	20	20	120
C	10	110	

10. In May 1995 the 10s of May 2003 offered a semiannually compounded yield of 6.65 percent. Recognizing that coupons are paid semiannually, calculate the bond's price.

Questions and Problems

1. Why might Fisher's theory about inflation and interest rates not be true?
2. You have estimated spot interest rates as follows:

Year	Spot rate
1	$r_1 = 5.00\%$
2	$r_2 = 5.40$
3	$r_3 = 5.70$
4	$r_4 = 5.90$
5	$r_5 = 6.00$

   (*a*) What are the discount factors for each date (that is, the present value of \$1 paid in year *t*)?
   (*b*) What are the forward rates for each period?
   (*c*) Calculate the PV of the following Treasury notes:
      (i) 5 percent, 2-year note
      (ii) 5 percent, 5-year note
      (iii) 10 percent, 5-year note
   (*d*) Explain intuitively why the yield to maturity on the 10 percent bond is less than that on the 5 percent bond.
   (*e*) What should be the yield to maturity on a 5-year zero coupon bond?
   (*f*) Show that the correct yield to maturity on a 5-year annuity is 5.75 percent.
   NO (*g*) Here is a harder question: Explain intuitively why the yield on the 5-year Treasury notes described in part (*c*) must lie between the yield on a 5-year zero coupon bond and a 5-year annuity.
3. Look at the spot interest rates shown in problem 2. Suppose that someone told you that the 6-year spot interest rate was 4.80 percent. Why would you not

believe him? How could you make money if he was right? What is the minimum sensible value for the 6-year spot rate?

4. Look just one more time at the spot interest rates shown in problem 2. What can you deduce about the 1-year spot interest rate in 4 years if
   (a) The expectations theory of term structure is right?
   (b) The liquidity-preference theory of term structure is right?
   (c) The term structure contains an inflation uncertainty premium?

5. Assume the term structure of interest rates is upward-sloping. How would you respond to the following comment? "The present term structure of interest rates makes short-term debt more attractive to corporate treasurers. Firms should avoid new long-term debt issues."

6. It has been suggested that the Fisher theory is a tautology. If the real rate of interest is defined as the difference between the nominal rate and the expected inflation rate, then the nominal rate *must* equal the real rate plus the expected inflation rate. In what sense is Fisher's theory *not* a tautology?

7. Look up prices of 10 U.S. Treasury bonds with different coupons and different maturities. Calculate how their prices would change if their yields to maturity increased by one percentage point. Are long-term bonds or short-term bonds most affected by the change in yields? Are high-coupon bonds or low-coupon bonds most affected?

*8. Look up prices of 10 corporate bonds with different coupons and maturities. Be sure to include some low-rated bonds on your list. Now estimate what these bonds would sell for if the United States government had guaranteed them. Calculate the value of the guarantee for each bond. Can you explain the differences between the 10 guarantee values?

9. Look in a recent issue of *The Wall Street Journal* at New York Stock Exchange bonds.
   (a) The yield shown in *The Wall Street Journal* is the current yield—that is, the coupon divided by price. Calculate the yield to maturity for a long-dated AT&T issue (assume annual interest payments).
   (b) How much higher is the yield on the AT&T bond than the yield on a Treasury bond with a similar maturity?
   (c) Glance quickly through the list of bonds and find one with a very high current yield. Calculate the yield to maturity on this bond.
   (d) Why is the yield to maturity on this bond so high?
   (e) Would the expected return be more or less than the yield to maturity?

10. Under what conditions can the expected real interest rate be negative?

11. Bond-rating services usually charge corporations for rating their bonds.
   (a) Why do they do this, rather than charge those investors who use the information?
   (b) Why will a company pay to have its bonds rated even when it knows that the service is likely to assign a below-average rating?
   (c) A few companies are not willing to pay for their bonds to be rated. What can investors deduce about the quality of these bonds?

*12. A 6 percent, 6-year bond yields 12 percent and a 10 percent, 6-year bond yields 8 percent. Calculate the 6-year spot rate. (Assume annual coupon payments.)

*13. In July 1989, stripped U.S. Treasury bonds were priced as follows:

Maturity	Price
1990	92.8%
1991	86.4
1992	80.3
1993	74.6
1994	69.2
1995	64.1
1996	59.3

Stripped bonds are "zero coupon" securities which make only one payment at maturity.
   (a) Estimate the spot rates of interest.
   (b) Estimate the forward rates of interest.
   (c) If the expectations theory of term structure is correct, what was the expected 1-year rate of interest in 1995?

14. Are high-coupon bonds more likely to yield more than low-coupon bonds when the term structure is upward-sloping or when it is downward-sloping?

*15. Look back to the first Backwoods Chemical example at the start of Section 23-5. Suppose that the firm's book balance sheet is:

Backwoods Chemical Company (Book Values)

Net working capital	$ 400	$1,000	Debt
Net fixed assets	1,600	1,000	Equity (net worth)
Total assets	$2,000	$2,000	Total liabilities and net worth

The debt has a 1-year maturity and a promised interest rate of 9 percent. Thus, the promised payment to Backwoods's creditors is $1090. The market value of the assets is $1200, and the standard deviation of asset value is 45 percent per year. The risk-free interest rate is 9 percent. Use Appendix Table 6 to value the Backwoods debt and equity.

16. Assume that Figure 23-3 correctly describes the path of short-term interest rates. The prices and yields of the three bonds that we described in Section 23-4 have changed. The yield to maturity of Medium is 8.18 percent. The yield to maturity on Long is currently 7.80 percent. Next period it will fall to 3.98 percent if the short-term interest rate falls and rise to 11.84 percent if the short-term rate rises.
   (a) Are these yields consistent, or are there arbitrage profits to be made?
   (b) How would your answer change if the yield of Medium were currently 7.93 percent and all other yields were unchanged?

17. In Section 23-3, we stated that the duration of the 8s of 2001 was 5.031 years. Construct a table like Table 23-3 to show that this is so.

*18. Look back at the example of the Short, Medium, and Long bonds in Section 23-4. Assume that to invest in Medium, investors require an expected return of 11 percent in year 1 (i.e., a risk premium of 3 percent).

Solutions to Problem Set 4

PROBLEM SET 4

[1] Suppose that in a certain economy we have that

$$f(t,T) > r_T$$

Our arbitrage strategy is as follows:

at period t → short sell a T-maturity zero bond at $P(t,T)$
↘ buy $[1+f(t,T)]$ $(T+1)$-maturity zero bonds at $P(t,T+1)$.

Since $[1+f(t,T)]P(t,T+1) = P(t,T)$ (see eq (3)), the above strategy involves zero investment, i.e. the net position at period t is:

$$[1+f(t,T)]P(t,T+1) - P(t,T) = 0$$

Next, at period T we borrow \$1 which we pay to the owner of the T-maturity zero (due to our short-selling above)

Thus at period $(T+1)$ → we pay back our loan $= 1+r_T$
↘ we receive $[1+f(t,T)]$ (recall that at period t we had bought $(T+1)$-maturity zeros)

Since $f(t,T) > r_T$ an arbitrage profit is possible.

⟷

Ex. [2], [3], [4], [5] see Lecture Notes I.

B&M, ch. 23 [Ignore exercise 16]

$\left\{\begin{matrix} \text{Notation in B\&M:} & r_2 \\ \text{Notation in LN I:} & Y(t,t+2) \end{matrix}\right\}$

[2] (a) Discount factors:

$$\frac{1}{[1+Y(t,t+T)]^T}, \quad T=1,2,3,4,5$$

i.e $\frac{1}{1+0.05}=0.952$, $\frac{1}{(1+0.054)^2}=0.9$

$\frac{1}{(1+0.057)^3}=0.847$, $\frac{1}{(1+0.059)^4}=0.795$, $\frac{1}{(1+0.06)^5}=0.747$

(b) We compute the forward rates by using eq. (5b):

$$[1+Y(t,T)]^{T-t}=(1+r_t)[1+f(t,t+1)]\cdots[1+f(t,T-1)]$$

i.e. $[1+0.054]^2=(1+0.05)(1+f(0,1)] \Rightarrow$

$\rightarrow$ Year 2: $f_2 \equiv f(0,1)=\frac{(1+0.054)^2}{1+0.05}-1=\underline{\underline{0.058}}$,

$$[1+0.057]^3=(1+0.05)[1+f(0,1)][1+f(0,2)]$$
$$=(1+0.05)\frac{[1+0.054]^2}{(1+0.05)}[1+f(0,2)] \Rightarrow$$

$\rightarrow$ Year 3: $f_3 \equiv f(0,2)=\frac{[1+0.057]^3}{[1+0.054]^2}-1=\underline{\underline{0.063}}$,

$$[1+0.059]^4 = (1+0.05)[1+f(0,1)][1+f(0,2)][1+f(0,3)]$$

$$= (1+0.05)\frac{[1+0.054)^2}{(1+0.05)}\frac{(1+0.057)^3}{(1+0.054)^2}[1+f(0,3)] \Rightarrow$$

$\rightarrow$ Year 4: $f_4 \equiv f(0,3) = \dfrac{(1+0.059)^4}{(1+0.057)^3} - 1 = \underline{\underline{0.065}}$,

$$[1+0.06]^5 = (1+0.05)[1+f(0,1)][1+f(0,2)][1+f(0,3)][1+f(0,4)]$$

$$= (1+0.05)\frac{(1+0.054)^2}{(1+0.05)}\frac{(1+0.057)^3}{(1+0.054)}\frac{(1+0.059)^4}{(1+0.057)^3}[1+f(0,4)] \Rightarrow$$

$\rightarrow$ Year 5: $f_5 \equiv f(0,4) = \dfrac{(1+0.06)^5}{(1+0.059)^4} - 1 = \underline{\underline{0.064}}$

(C) Assume annual coupon payments and \$1000 face value.

(i) 5% 2-year note

$$PV = \frac{50}{1.05} + \frac{1050}{(1.054)^2} = 992.79$$

(ii) 5% 5-year note

$$PV = \frac{50}{1.05} + \frac{50}{(1.054)^2} + \frac{50}{(1+0.057)^3} + \frac{50}{(1.059)^4} + \frac{1050}{(1.06)^5} = 959.34$$

(iii) 10% 5-year note

$$PV = \frac{100}{1.05} + \frac{100}{(1.054)^2} + \frac{100}{(1.057)^3} + \frac{100}{(1.059)^4} + \frac{1100}{(1.06)^5} = 1171.43$$

[Note: we are using eq. (5a)]

(d) The 10% bond has more of its total payments coming earlier when interest rates are low, than does the 5% bond.

(e) The yield to maturity on a 5-year zero should be

$$r_5 \equiv y(t, t+5) = 6\%$$

(f)

$$\frac{1}{(1+0.05)} + \frac{1}{(1+0.054)^2} + \frac{1}{(1+0.057)^3} + \frac{1}{(1+0.059)^4} + \frac{1}{(1+0.06)^5} =$$

$$= \left[\frac{1}{r} - \frac{1}{r(1+r)^5} \right]$$

$\Rightarrow$ yield to maturity on a 5-year annuity : $r = 5.75\%$

[3] A six-year spot rate of 4.8% implies a negative forward rate:

year 6: $f_6 \equiv f(0,5) = \dfrac{(1+0.048)^6}{(1+0.06)^5} - 1 = -0.01$

There is an arbitrage opportunity:

borrow \$1000 for 6 years at 4.8%
lend \$990 for 5 years

So $1000\,(1+0.048)^6 = 990\,(1+0.06)^5$

and in addition you have \$10 to spend now.

[4] Under the UEH, the expected spot rate equal the forward rate:

(a)

$$E(r_T) = f(t, T)$$

So the expected 1-year spot in 4 years is

$$_4r_5 \equiv E(r_4) = f(0,4) = \frac{(1+0.06)^5}{(1+0.059)^4} - 1 = 0.064$$

(b) If the liquidity preference theory is correct, the expected spot rate will be less than 6.4%

(c) If the term structure contains an inflation uncertainty premium, the exp. spot should be less than 6.4%

[5] It may be upward sloping because short-term rates are expected to rise or because long-term bonds are more risky (either investors want to lend short-term or inflation is uncertain). A sensible starting position is to assume that all debt is fairly priced.

[11] (a) Information is difficult to charge for; you cannot stop one person from transmitting it to another free. The bond-rating services thus find it much easier to charge the companies, rather than the investors.

(b) Suppose no bonds were rated. Companies with the highest quality bonds want to demonstrate that fact. Once the highest quality bonds have been rated, companies with the highest quality bonds of those remaining have an incentive to demonstrate that their bonds are the best of the remainder. And so on ..... until only the lowest quality bonds are left. So companies are willing to pay to have their bonds rated to alleviate investors' fears that the bonds might be of even lower quality.

(c) It follows from (b) that only companies with extremely poor quality bonds will not pay to have them rated.

[12]

Investment	Yield	C_1	...	C_5	C_6	Price
6% bond	12%	60	...	60	1060	753.32
10% bond	8%	100	...	100	1100	1092.46.

We need to find a combination of these two bonds that has a cash flow only at $t=6$. We can achieve this by buying a one 6% bond and selling the equivalent of 60 per cent of a 10% bond. This portfolio costs

$$753.32 - 0.6\,(1092.46) = 97.84$$ and

has a zero cash flow for years 1 through 5. At $t=6$ the portfolio's cash flow is

$$C_6 = 1060 - 0.6(1100) = 400$$

Thus $$97.84 = \frac{400}{[1+Y(0,6)]^6}$$

$$\left(\text{or } 97.84 = \frac{400}{(1+R_6)^6} \right)$$

So the 6-year spot rate is 26.5%

[13] (a) Year 1999 : $t=0$, T-year maturity

We use the following equation:

$$P(0,T) = \frac{100}{[1+Y(0,T)]^T} \Rightarrow$$

$$[1+Y(0,T)]^T = \frac{100}{P(0,T)} \quad , \quad T = 1, 2, \ldots, 7$$

$\Rightarrow r_1 \equiv Y(0,1) = 7.76\%$ $\quad r_5 \equiv Y(0,5) = 7.64\%$
$r_2 \equiv Y(0,2) = 7.58\%$ $\quad r_6 \equiv Y(0,6) = 7.69\%$
$r_3 \equiv Y(0,3) = 7.59\%$ $\quad r_7 \equiv Y(0,7) = 7.75\%$
$r_4 \equiv Y(0,4) = 7.60\%$

(b) We compute the forward rates in a similar way to this in exercise [2]:

$$f_2 \equiv f(0,1) = \frac{(1+0.0758)^2}{1+0.0776} - 1 \Rightarrow f_2 = 7.4\%$$

$$f_3 = \frac{(1+0.0759)^3}{(1+0.0758)^2} - 1 = 7.6\%$$

$$f_4 = \frac{(1+0.076)^4}{(1+0.0759)^3} - 1 = 7.64\%$$

. . . .

$$f_5 = 7.8\%$$
$$f_6 = 7.96\%$$
$$f_7 = 8.09\%$$

(c) If the UEH is true then exp. 1-year spot = forward
So exp. spot in 1995 was 8.09%.